# AstroChallenge 2020
# Senior Team Round

**SOLUTIONS**

Saturday 5th December 2020

**PLEASE READ THESE INSTRUCTIONS CAREFULLY.**

1. This paper consists of **29** printed pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your name, school, and team.
6. It is your responsibility to ensure that your answer script has been submitted.
7. The marks for each question are given in brackets in the right margin, like such: [**2**].
8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

# Question 1 Dubious Statements

This question comprises 8 statements. For each statement, indicate clearly whether it is **TRUE** or **FALSE**.

Support your answer with no more than 3 sentences, including any assumptions where required. You may draw up to one additional diagram if they aid your explanation.

**Mathematical working is not required, and there are no errors in any of the statements below.**

Each statement is worth $2\frac{1}{2}$ marks, **attributed only to the quality of the justification**.

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(a) Total and annular solar eclipses cannot be observed from either of the Earth's geographic poles. **[$2\frac{1}{2}$]**

**Solution:**

False.

**Explanation 1:** Since the Moon's orbit is inclined at 5° relative to the ecliptic plane, total eclipses are observed at the poles only near the ascending and descending nodes when the Sun is above the local horizon. A total eclipse has occurred at the North Pole on 19 March 2015, and will occur at the South Pole on 16 January 2094.

(b) It is possible for the dust and ion tails of a comet that has entered the inner Solar System to be oriented perpendicularly in space. **[$2\frac{1}{2}$]**

**Solution:**

THIS QUESTION CAN BE ANSWERED IN BOTH WAYS.

True.

**Explanation 1:** The ion tail of a comet points directly away from the Sun as the ions follows the outward direction of the magnetic field of the plasma in the solar wind. The dust tail generally follows the comet's orbital trajectory (discounting radiation pressure) and thus the (start of the) tails are orthogonal at perihelion.

**Explanation 2:** The ion tail of a comet points directly away from the Sun as the ions follows the outward direction of the magnetic field of the plasma in the solar wind. The dust tail comprises dust particles which are affected by solar radiation pressure, and thus may curve towards the orbital trajectory near the semi-latus rectum, making it orthogonal to the ion tail.

False.

**Explanation 1:** The ion tail of a comet points directly away from the Sun as the ions follows the outward direction of the magnetic field of the plasma in the solar wind. However, the dust tail generally points away from the Sun as the dust particles are affected by solar radiation pressure, hence the two tails cannot be orthogonally oriented.

(c) Elements with atomic numbers greater than that of nickel are only formed in stars during highly energetic events. **[$2\frac{1}{2}$]**

**Solution:**

False.

**Explanation 1:** The s-process allows the formation of atomic nuclei heavier than iron via neutron capture and subsequent slow beta-decay in the interior of late-stage (primarily AGB) stars.

(d) Hydrogen in some form is the dominant interstellar component in both emission and reflection nebulae, and account for the differences in their appearance. **[2½]**

**Solution:**

False.

**Explanation 1:** Interstellar gas is indeed predominantly molecular or atomic hydrogen. In emission nebulae, hot massive stars produce sufficient high-energy UV photons to cause significant ionisation of molecular hydrogen, giving emission nebulae their characteristic red colour due to strong emission from the H$\alpha$ line. In reflection nebulae, however, the energy from nearby stars is insufficient to ionise H but sufficient to render dust clouds visible by scattering, giving reflection nebulae a bluish colour due to greater scattering of shorter wavelengths (Rayleigh scattering).

(e) The energy required to power the electromagnetic beams of a pulsar along its magnetic poles results in its observed period increasing continuously. **[2½]**

**Solution:**

THIS QUESTION CAN BE ANSWERED IN BOTH WAYS.

True.

**Explanation 1:** The fast rotation generates an electrical field from the motion of the magnetic field, which accelerates charged particles creating the pulsar's beams. By conservation of energy, the rotational energy of the (rotation-powered) pulsar decreases due to emission of the beams, slowing the pulsar's rotation (spin-down). This assumes the absence of external forces and that the moment of inertia of such pulsars remains unchanged.

False.

**Explanation 1:** The fast rotation generates an electrical field from the motion of the magnetic field, which accelerates charged particles creating the pulsar's beams. Rotation-powered pulsars however do experience inherent glitches (temporary decreases in rotation period) that interrupt the monotonic spin-down. This occurs due to internal processes that transfer angular momentum from the faster-spinning superfluid core to the (decoupled) outer crust.

**Explanation 2:** The fast rotation generates an electrical field from the motion of the magnetic field, which accelerates charged particles creating the pulsar's beams. However, huge stresses on the surface of the pulsar due to the twisting on the strong magnetic field lines may result in neutron starquakes, where the crust adjusts itself closer to a shape closer to a perfect sphere. This decreases the moment of inertia of a pulsar and results in a persistent decrease of the observed period above that predicted by pulsar spin-down.

**Explanation 3:** The fast rotation generates an electrical field from the motion of the magnetic field, which accelerates charged particles creating the pulsar's beams. However, not all pulsars are rotation-powered and exhibit a monotonic spin-down, as some pulsars are powered by accretion of matter from a companion star in a binary system. For accretion-powered pulsars, the gravitational potential energy of accreted matter powers the electromagnetic beams, and these pulsars exhibit irregular spin-up and spin-down behaviour depending on the rate of accretion.

(f) The distribution of bodies of differing masses (at different radii) in a gravitationally bound system remains unchanged with time in the absence of external forces. **[2½]**

**Solution:**

False.

**Explanation 1:** Even though such a system is closed, cluster members (stars/galaxies) both evolve and also gravitationally interact in a chaotic manner. Due to the equipartition of kinetic energy during gravitational interaction of cluster members, over long timescales (the relaxation time), dynamical mass segregation occurs where high-mass members sink towards the cluster's center, whereas low-mass members are promoted to higher orbits. Some low-mass cluster members may be entirely ejected from the system in gravitational interactions with high-mass members (cluster evaporation).

**Explanation 2:** Even though such a system is closed, cluster members (stars/galaxies) both evolve and also gravitationally interact in a chaotic manner. For star clusters, high-mass members will evolve more quickly and reach their end-of-life phases, hence over long timescales, the overall distribution of bodies of differing masses will increasingly be dominated by low-mass cluster members across the entire radii of the system.

(g) Variations in the surface brightness of an observed galaxy across its cross-sectional area may be used to refine the accuracy of a distance estimate derived from a standard candle. **[2½]**

**Solution:**

True.

**Explanation 1:** This is the surface brightness fluctuation (SBF) distance indicator and is a statistical method relying on the non-uniformity of galaxy brightness profiles. Whilst the mean surface brightness of a galaxy (in absence of extinction) remains largely unchanged with distance, distant galaxies have a smaller (cross-sectional) angular area and thus the mean number of stars in each angular area is greater. By subtracting a smooth profile of the galaxy from the raw pixel data, we obtain residual noise whose r.m.s variation is smaller for more distant galaxies due to the averaging of the distribution of stars across the angular area.

(h) For an astrophotography set-up with a Newtonian telescope, removing the telescope tube in favour of an open truss design improves image contrast at the expense of image stability. **[2½]**

**Solution:**

THIS QUESTION CAN BE ANSWERED IN BOTH WAYS.

True.

**Explanation 1:** Replacing a poorly baffled/flocked tube with an open truss design (with light shroud) can improve image contrast by eliminating internal reflections of stray light in the tube. However, the truss tube may flex under the weight of the secondary mirror/focuser/eyepiece/camera, resulting in miscollimation of the telescope, reducing image stability.

(Note: For 'True' responses, you must explain **both** how the contrast improves and how the stability worsens.)

False.

**Explanation 1:** An open truss design increases the amount of stray light collected (if no light shroud is used), increasing the background brightness and consequently reducing the contrast of stars/DSOs relative to the background sky.

**Explanation 2:** An open truss design increases the amount of ambient air to the mirrors, allowing it to reach thermal equilibrium with the surroundings quicker. It also eliminates tube currents, improving stability.

**Explanation 3:** An open truss design reduces the OTA's weight, reducing strain on a motorised mount. This improves overall tracking accuracy and hence results in more stable images.

# Question 2 Brightest Day, Blackest Night

*In brightest day, in blackest night,*
*No evil shall escape my sight.*
*Let those who worship evil's might*
*Beware my power: Green Lantern's light!*

–Oath of the Green Lantern Corps
(Credit: DC Comics)

## Introduction

The oath of the Green Lantern Corps is the central theme to this question. There are four parts to this question, mostly dealing with certain intricacies of the astronomically-relevant words in the oath: Brightest Day, Blackest Night, and Green Lantern's Light.

Do your best to defeat evil's (read: the question's) might!

**FOR THE ENTIRETY OF THIS QUESTION, ASSUME ALL PLANETS ARE SPHERICAL AND HAVE CIRCULAR ORBITS.**

## Part I Brightest Day

For this part, we only consider the Arctic circle, but with understanding that similar conclusions apply for the Antarctic circle as well. Collectively they are called the polar circles.

Polar day (or Midnight Sun) is a phenomenon whereby the centre of the Sun is visible at all times of the day. For the purposes of this question, the edge case of 'exactly on the horizon' is treated as being visible. The definition of the Arctic circle relies on this phenomenon and therefore (in theory) the circle is not fixed over time. The following presents two definitions of the Arctic circle.

**Definition 1.** The Arctic circle is the southernmost latitude above the equator from which the centre of the Sun can be seen at all times throughout the day at least once a year, at sea level and ignoring the effect of atmosphere.

**Definition 2.** The Arctic circle is the latitude defined by

$$\text{Latitude of Arctic Circle} = 90^\circ - \text{Axial tilt of Earth.}$$

(Note: Since the axial tilt varies slightly, the value commonly quoted is typically an average over several years. For the purposes of this question, however, you should ignore this averaging.)

(a) (i) With the aid of a **clear** diagram, justify the equivalence of both definitions, i.e. show that Definition 2 can be deduced from Definition 1, and vice-versa. Your diagram should be well-labelled. **[2]**

**Solution:**

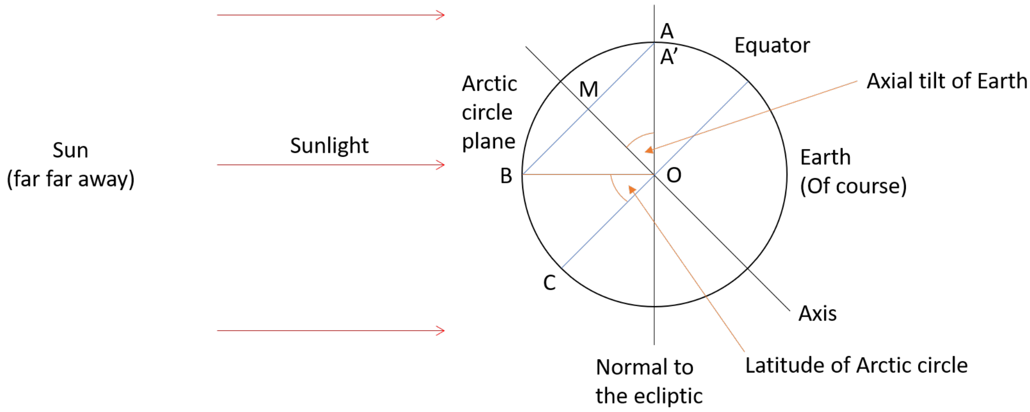



**Figure 1:** Cross-sectional diagram of Earth at summer solstice. $O$ is the centre of the Earth. $A$ is the intersection of axis with the Earth's surface. $A'$ and $B$ are the intersections of Arctic circle plane with Earth's surface. $M$ is the intersection of chord $A'B$ with the axis. Clearly $\angle BMO = \angle A'MO = 90°$.

The important observation is that the Arctic circle being the southernmost latitude which sees the Sun at all times throughout the day at least once a year, is equivalent to the condition that $A = A'$.

($1 \implies 2$): By Definition 1, $A = A'$. Note that $\Delta AOB$ is isosceles since $AO = OB$. Since $\angle AMO = 90°$, $OM$ is a perpendicular bisector of $\angle AOB$. Hence $\angle MOB = \angle AOM$. Finally, since the axis and the equatorial plane are perpendicular, $\angle MOC = 90°$, so that $\angle BOC = 90° - \angle MOB = 90° - \angle AOM$, as needed.

($2 \implies 1$): We are to show $A' = A$.

Since $\angle BOC = 90° - \angle AOM$ by Definition 2, and since $\angle MOC = 90°$, $\angle MOB = \angle MOC - \angle BOC = \angle AOM$. Note now that $BO = AO$ as both are radii of Earth, and that $MO$ is common to both $\Delta AOM$ and $\Delta BOM$. By the SAS property of congruent triangles, $\Delta AOM \cong \Delta BOM$.

On the other hand, recall that $\angle BMO = \angle A'MO = 90°$. Notice that using the property that the axis (diameter) bisects the chord $A'B$, we have $MB = MA'$. Furthermore $BO = A'O$. By the RHS property of right-angled triangle congruence, $\Delta A'OM \cong \Delta BOM$.

Now since $\Delta A'OM \cong \Delta AOM$ and both $A$ and $A'$ lie to the right of $M$, therefore $A' = A$, as needed.

(ii) How many times does the Sun set at the (axial) poles each year? Explain why. If necessary, you may wish to draw or reuse the above answer's diagram. **[1]**

**Solution:**

Once. At the axial poles the Sun does not shift throughout the sky due to Earth's rotation. Its movement is merely due to Earth's orbit. As a consequence, the Sun crosses the horizon if and only if the pole crosses from polar day to polar night. This happens at the equinoxes, and the Sun thus rises once and sets once per year.

(iii) By definition, at the Arctic circle, polar day should occur exactly once per year on a certain day. What is this day known as? **[1]**

**Solution:**

Summer solstice/June solstice.

Note: The solstice must be mentioned. No credit is awarded for specific date(s).

In reality, however, to an observer at sea level, the phenomenon of polar day occurs up to approximately 90 km south of the Arctic circle. The main reason for this phenomenon is *atmospheric refraction*.

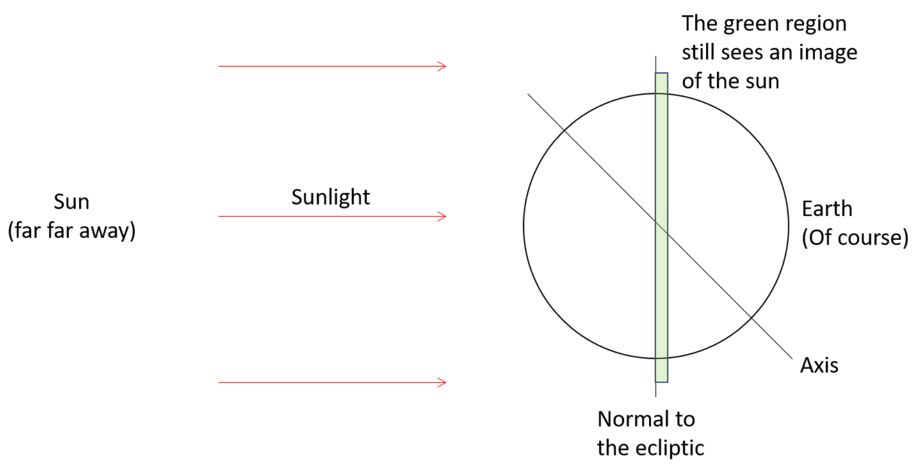
(b) (i) What is atmospheric refraction? [½]

**Solution:**

It is the deviation of light from a straight line of travel as it enters and passes through the atmosphere. This effect is caused by refraction, hence the name, due to the variation of air density with height in the atmosphere.

(ii) Explain briefly how atmospheric refraction gives rise to this real-world observation. You may wish to draw a suitable diagram to assist. **[1½]**

**Solution:**




Due to the bending of light as it passes through the atmosphere, light from the Sun reaches just slightly beyond the vertical to the ecliptic.

Consequently, this means that an observer on Earth in this zone can observe an image of the (centre of the) Sun, despite the true position of the (centre of the) Sun being just below the horizon. As such, polar day ends up being observed for a short distance below the Arctic circle, since polar day relies on the *observation* of the centre of the Sun, and not its true positioning in space.

## Part II Blackest Night

The counterpart to the polar day is the polar night, where the Sun is never visible for the entire day. The duration of polar night is defined as the total continuous duration where the Sun never rises above the horizon (for an observer at sea level). The duration of polar day is similarly defined.

(c) As mentioned above, polar day can occur outside (i.e. at latitudes closer to the equator than) the polar circles. Answer the following two questions.

(i) Can the polar night ever occur outside (or on) the polar circles? Explain. **[1]**

**Solution:**

No. On the polar circles, in theory the centre of the Sun should stay *at* the horizon on exactly one day. Due to atmospheric refraction, however, on this day the centre of the Sun will appear above the horizon. Polar night thus does not occur.

Outside the polar circles, the Sun will not fall below the horizon for more than a day, even without considering refraction.

(ii) At a given latitude, which is more: the number of polar days, or the number of polar nights? Or are they the same? Explain. [1½]

**Solution:**

At latitudes where polar days are not experienced, they are the same: zero. Any region which experiences polar night will experience polar day.

Otherwise, polar days are more. Due to atmospheric refraction, the time at which the latitude circle is completely within the 'day zone' (i.e. left side in (iv)(b) including green zone) is increased. At the same time, the time it is completely within the 'night zone' (right side excluding green zone) is decreased. As a result, this causes the number of polar days to be higher than the number of polar nights.

Note: We accept $\geq$ instead of $>$ for the second half, with appropriate modifications to the argument as necessary. Answers applicable only to polar latitudes are capped at a score of 1.

## Part III No (Part of) E.V.I.L. Shall Escape My S.I.G.H.T.

**NOTE: IN THIS PART, WE ONLY CONSIDER NORTHERN LATITUDES.**

A certain scientific villain wants to create a base on the airless planet E.V.I.L. (Extremely Very Irritating Lair, because which self-respecting villain doesn't have a whole planet as an evil lair that's very irritating to infiltrate?). This planet orbits a (fictional) star S.I.G.H.T. (Super Irritating Ginormous Hot Thing, because that's totally how you explain a star badly).

E.V.I.L.'s axial tilt is $\varphi$, its orbital period is $T$, and its rotational period is $p$. You may assume that $T \gg p$. Its radius is $R$. At time 0, the planet's axis forms an angle $\frac{\pi}{2} - \varphi$ with respect to the planet-star line, with North tilted towards S.I.G.H.T.

At time $t$, the angle $\tau$ between E.V.I.L.'s axis and the planet-star line is given by

$$\tau = \cos^{-1}\left(\sin\varphi \cos\frac{2\pi t}{T}\right), \qquad \frac{\pi}{2} - \varphi \leq \tau \leq \frac{\pi}{2} + \varphi.$$

Note that an angle above $\frac{\pi}{2}$ represents the Northern hemisphere tilted away from S.I.G.H.T.

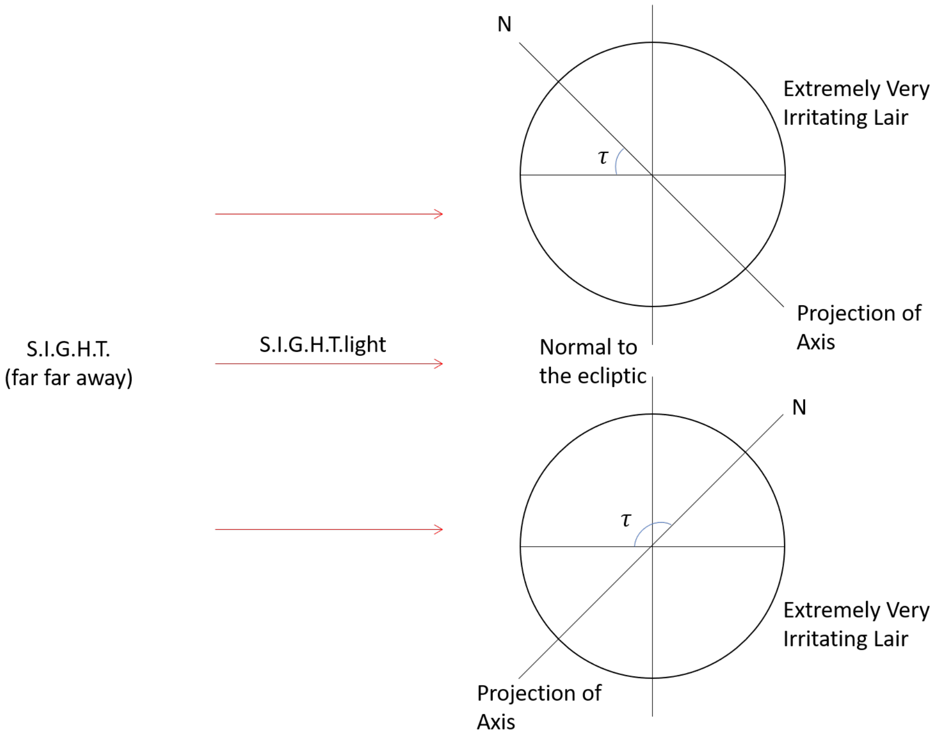



**Figure 2:** Cross-sectional diagram representation of $\tau$ at two different times. Note that the tilt $\tau$ of the projection of axis line changes with $t$. When $t = 0$, $\tau = \frac{\pi}{2} - \varphi$.

The goal of this part is to derive an expression for length of time spent in direct S.I.G.H.T.light. Or if you prefer, length of daylight.

(d) **(BONUS)** Prove the given expression for $\tau$. **[2]**

**Solution:**

If $\varphi = 0$, the result is obvious.

Assume $\varphi \neq 0$. Construct a coordinate system with the star at $(0, 0, 0)$ and the planet at $(1, 0, 0)$ when the angle is $\varphi$ (note that the Earth-Sun analogue is the summer solstice). A vector describing the direction of axial tilt at this point is $v = (0, 0, \frac{1}{\tan\varphi}) - (1, 0, 0) = (-1, 0, \frac{1}{\tan\varphi})$.

The angle swept by the planet in time $t$ is $\frac{2\pi t}{T}$ if the planet travels counter-clockwise, and $-\frac{2\pi t}{T}$ if the planet travels clockwise. In the first case after time $t$ the planet will lie at $u = (\cos\frac{2\pi t}{T}, \sin\frac{2\pi t}{T}, 0)$. In the second case it will lie at $u = (\cos\frac{2\pi t}{T}, -\sin\frac{2\pi t}{T}, 0)$. In either case,

$$\cos\tau = \frac{(0 - u) \cdot v}{\|0 - u\| \|v\|} = \frac{\cos\frac{2\pi t}{T}}{\sqrt{1 + \frac{1}{\tan^2\varphi}}}.$$

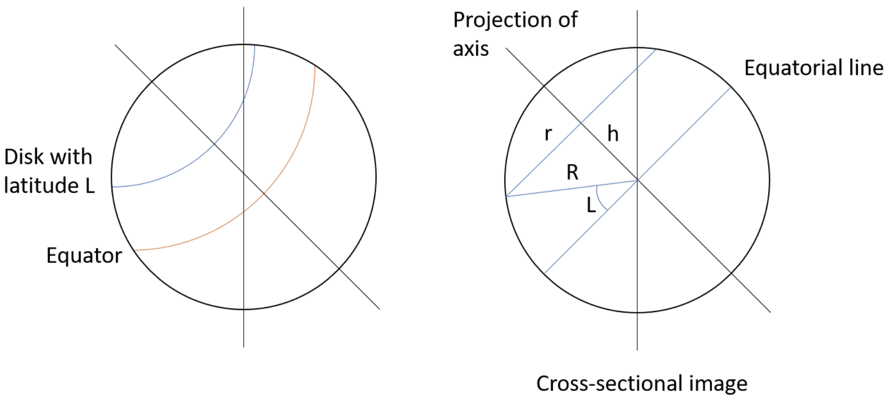
Recall the equation $1 + \cot^2 x = \csc^2 x$. The denominator in the above is thus $\sqrt{\csc^2\varphi} = \csc\varphi = \frac{1}{\sin\varphi}$, where the modulus disappears since $\sin\varphi \geq 0$ for all possible values of $\varphi$. The result follows.

(e) Let $L$ be a given (fixed) northern latitude. The points on E.V.I.L. with latitude $L$ form a circle around E.V.I.L. This circle has radius $r$ and is at a distance $h$ from the equatorial plane. With a diagram, explain why we have

$$h = R\sin L \qquad \text{and} \qquad r = R\cos L.$$

**[1]**

**Solution:**




The cross-sectional image cuts through the centre of the Earth. We have by very basic trigonometry,

$$h = R \sin L, \qquad r = R \cos L.$$

(f) Let $\mathscr{D}$ be the disk on E.V.I.L. formed by the circle in part **e**. Let $\ell$ be the normal line to the ecliptic that passes through the centre of E.V.I.L. Notice that depending on $\tau$ and $L$, $\ell$ may not intersect $\mathscr{D}$.

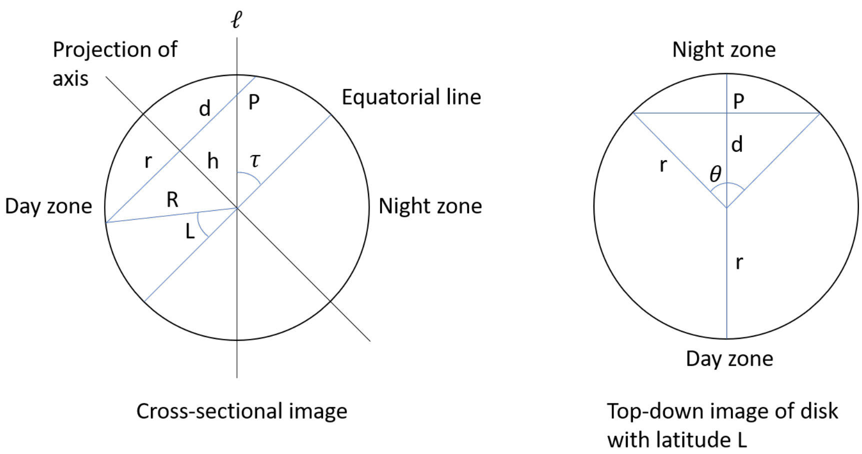
Suppose for the moment that $\ell$ intersects $\mathscr{D}$ at a point $P$.

(i) Let $d$ be the distance from the centre of $\mathscr{D}$ to $P$. Show that

$$d = R \sin L \cot \tau.$$

Again, $d < 0$ means that the Northern hemisphere is tilted away from S.I.G.H.T. **[1]**

**Solution:**




Again, the image makes it clear. Since the tilt with respect to $\ell$ is $\frac{\pi}{2} - \tau$, this value is simply

$$d = h \tan\left(\frac{\pi}{2} - \tau\right) = R \sin L \cot \tau.$$

Alternatively, by alternate angles one has $\frac{h}{d} = \tan \tau$.

(ii) Hence or otherwise show that at latitude $L$ and position at time $t$, the length of time $D$ in direct S.I.G.H.T.light is

$$D = \frac{p\cos^{-1}(-\tan L\cot\tau)}{\pi},$$

where the inverse cosine is taken over $[0, \pi]$.

(Hint: Use the fact that $\pi - \cos^{-1} x = \cos^{-1}(-x)$.)[1] **[1]**

**Solution:**

We determine the fraction of the disk in the day zone, a.k.a. the fraction of the circle made up by the major arc below the topmost horizontal line in the top-down image. The angle $\theta$ at centre of circle formed by the isosceles triangle is

$$\theta = 2\cos^{-1}\frac{d}{r} = 2\cos^{-1}\frac{R\sin L\cot\tau}{R\cos L} = 2\cos^{-1}(\tan L\cot\tau).$$

Therefore the fraction made up by the major arc is

$$\frac{2\pi-\theta}{2\pi} = \frac{2\pi - 2\cos^{-1}(\tan L\cot\tau)}{2\pi} = \frac{\pi-\cos^{-1}(\tan L\cot\tau)}{\pi} = \frac{\cos^{-1}(-\tan L\cot\tau)}{\pi}.$$

The length of day is therefore

$$D = \frac{p\cos^{-1}(-\tan L\cot\tau)}{\pi}.$$

(g) Now we consider what happens when $\ell$ does not intersect $\mathscr{D}$. This only occurs for latitudes $L > \frac{\pi}{2} - \varphi$. In this part, for simplicity consider only $0 \le t \le T$.

(i) Show that polar day occurs at a fixed latitude $L > \frac{\pi}{2} - \varphi$ when

$$0 \le t \le Q \qquad \text{or} \qquad T - Q \le t \le T,$$

where $Q = \frac{T}{2\pi}\cos^{-1}(\csc\varphi\cos L)$. Here the inverse cosine is taken over $\left[0, \frac{\pi}{2}\right]$. **[1]**

**Solution:**

Despite the scary looking nature of the question, this is just a standard exercise in trigonometry. For time $t$ and latitude $L$, polar day occurs if

$$L \ge \tau \qquad \Longrightarrow \qquad L \ge \cos^{-1}\left(\sin\varphi\cos\frac{2\pi t}{T}\right).$$

Then,

$$\sin\varphi\cos\frac{2\pi t}{T} \ge \cos L \qquad \Longrightarrow \qquad \cos\frac{2\pi t}{T} \ge \cos L\csc\varphi.$$

The RHS is a fixed value. The needed range follows from solving for $t$ in the usual manner.

(ii) The same logic above (or an even simpler argument) can be used to derive a similar range of $t$ during which polar night occurs.

Hence or otherwise, copy and complete the following expression for the length of day $D$ at time $t$, in terms

[1] For the math buffs. From $\cos(\pi-\theta) = -\cos\theta$, by writing $\theta = \cos^{-1} x$ in $[0,\pi]$, we get $\cos(\pi - \cos^{-1} x) = -x$. Taking $\cos^{-1}$ in $[0,\pi]$ on both sides gives $\pi - \cos^{-1} x = \cos^{-1}(-x)$. Magic! C'est magnifique!

of $p$, $L$, $\tau$, $T$, and $Q$:

$$D = \begin{cases} \_\_\_\_\_\_\_\_, & \text{if } L > \frac{\pi}{2} - \varphi \text{ and } (\_\_\_\_\_\_\_\_ \text{ or } \_\_\_\_\_\_\_\_), \\ \_\_\_\_\_\_\_\_, & \text{if } L > \frac{\pi}{2} - \varphi \text{ and } \_\_\_\_\_\_\_\_, \\ \_\_\_\_\_\_\_\_, & \text{else.} \end{cases}$$

(Note: Proof for the range of $t$ during polar night is not required.) **[1]**

**Solution:**

The range of polar night is $\frac{T}{2} - Q < t < \frac{T}{2} + Q$.

To prove this, although NOT REQUIRED, is pretty simple, just consider the open $2Q$ time interval centred around the winter solstice at $t = \frac{T}{2}$.

That done, we can just fill in the blanks.

$$D = \begin{cases} p, & \text{if } L > \frac{\pi}{2} - \varphi \text{ and } (0 \le t \le Q \text{ or } T - Q \le t \le T), \\ 0, & \text{if } L > \frac{\pi}{2} - \varphi \text{ and } \frac{T}{2} - Q < t < \frac{T}{2} + Q, \\ \frac{p \cos^{-1}(-\tan L \cot \tau)}{\pi}, & \text{else.} \end{cases}$$

(h) This certain scientific villain needs a place that experiences lots of time in direct S.I.G.H.T.light over one orbit. At what latitude(s) should he place his base? Or is his quest to find the best location a waste of time? Justify.

(Note: Remember that E.V.I.L. is airless.) **[1½]**

**Solution:**

It is a waste of time, unfortunately. At any point of the planet, the total amount of time spent in direct S.I.G.H.T.light will average out to be the same over an orbit - exactly $\frac{T}{2}$, to be precise.

One intuitive way is to view it using opposite points of the orbit. When the planet is at opposite points of the orbit, the portions of the latitude circle (at any chosen latitude) in 'day' and in 'night' are switched around. That is to say, the length of day at any point of the orbit is the length of night at the exact opposite point, and vice-versa. Since this holds for all points of the orbit, the average will turn out to be precisely half the orbital period, i.e. $\frac{T}{2}$.

Note: Arguing this fact completely mathematically/rigorously is not as easy, despite the intuitive nature of the question. We don't expect a full qualitative argument here, a brief logical reasoning will suffice.

## Part IV Green Lantern's Light

Sadly, outside of special effects, we will never see Green Lantern's light (although if you do, something bad is probably happening so you should run anyway). Thankfully, the Northern lights are here to provide a beautiful alternative! Commonly known as Aurora borealis, these lights are commonly seen in winter, spring, and autumn, inside and around the Arctic circle.

The lights are typically green, although they can appear in other varieties of colour. They typically form in an irregular oval (the aurora oval) centred roughly on the North magnetic pole. They are formed when particles, typically electrons and protons, from the solar wind interacts with particles in the atmosphere.

(j) (i) Near the Arctic circle, why are the lights not commonly seen in summer? **[1]**

**Solution:**

The lights are typically weak and they are overpowered by the bright summer nights near the Arctic circle. Remember that even though the Sun may be just below the horizon, twilight still occurs.

(ii) Explain the mechanism of formation of the Northern lights. **[4]**

**Solution:**

Earth has a protective magnetic layer (called the magnetosphere) which deflects incoming charged particles from the solar wind via the Lorentz force.

While most particles are deflected, however, with the correct angle of interaction (typically when the magnetic field is at an angle to the incoming particles), the deflection is less severe and instead of causing the particles to be deflected around the planet, instead causes the particles to spiral along the field lines instead into the Earth's atmosphere.

These particles collide with oxygen and nitrogen high in the Earth's atmosphere, exciting the gases. The low air densities allow for spontaneous de-excitations which would be otherwise impossible at lower altitudes.

These de-excitations cause the characteristic green (from oxygen) that we associate with aurorae. Other colours are from nitrogen, or in the case of red, also oxygen.

# Question 3 Astronomical Project – Galaxy Morphology

Ever wondered how professional astronomers tackle questions on the universe? In this question, we will attempt to explore that by role-playing as a team of professional astronomers! Note that it is always good to tackle a problem as a team as no one is an expert at everything. Some of us are better at some skills, or are more knowledgeable than others.

For our own AC2020 line-up of astronomers (QMs), we have:


Without further ado, let us start our journey as a team of aspiring astronomers working together to unveil the mysteries of the universe!

## Preamble

In view of large photometric surveys such as the Sloan Digital Sky Survey (SDSS), astronomers have been using a well-known correlation between galaxy colour and the galaxy's morphology to infer the morphological types of galaxies. The term 'morphology' is just a more sophisticated way of describing the 'type' of galaxy (i.e. spiral, elliptical, etc.).

In this project, we will be verifying the following claim:

> According to this colour-morphology relation, spiral galaxies are statistically bluer, disk-dominated, and hold more star formation than their elliptical counterparts.

## Part I Astronomical Preliminaries


First off, we will kick off the project with a quick refresher course from our local theoretical astrophysicist!

(a) The crux of this project involves investigating the colours of the galaxy.

(i) Briefly explain what causes the colour of different galaxies to be different. **[1]**

(ii) Using your answer in part (i), suggest a reason why spiral galaxies tend to appear bluer as compared to their elliptical counterparts. **[2]**

**Solution:**

Part (i): The colour of the galaxy is primarily determined by the colours of the stars that make up the galaxy.

Part (ii):

- Presence of spiral arms indicates active star formation. This causes the formation of young massive stars, which constitutes the blue colour observed in some spiral galaxies.
- Absence of spiral arms in elliptical galaxies indicates inactive star formation. The red colour results from the stars which are older and less massive and thus having a longer lifetime as compared to their heavier counterparts.

## Part II Data Collection and Image Processing


Next, we will have our amazing expert in practical astronomy, here to spearhead the collection of data! This is done by compiling galaxy images and data from various sources, such as the SDSS etc.

As suggested by our expert Grace, one quick way to classify if a galaxy is spiral or elliptical is by using the quantity $fracdeV$.

### Understanding $fracdeV$

If we look at the bulges of a spiral galaxy, or at large elliptical galaxies, the surface brightness $\mu(r)$, where $r$ represents the distance of a point located in the vicinity of the galaxy as measured from the centre of the galaxy, typically follows a $r^{\frac{1}{4}}$ law known as the de Vaucouleurs fit. This is described by the following equation:

$$\mu(r) = \mu_e + 8.3268\left(\left(\frac{r}{r_e}\right)^{\frac{1}{4}} - 1\right) \tag{1}$$

However, for galaxies more similar to disks, the intensity $I(r)$ of these galaxies is believed to follow an exponential decay:

$$I(r) = I_e \exp\left(-0.436\left(\frac{r}{r_e} - 1\right)\right) \tag{2}$$

Note that in equations (1) and (2), $I_e$, $\mu_e$, and $r_e$ are constants. They may or may not have physical interpretations. Also, note that $I(r_e) = I_e$.

When finding the best fit to model the light intensity profile of a galaxy, in almost all cases some combination of both curves above are required. The quantity $fracdeV$ represents the accuracy of describing the light intensity profile with only the de Vaucouleurs fit (i.e. with only equation (1)). It takes a value between 0 and 1 inclusive. A value of 1 represents a complete fit with equation (1), and 0 represents a complete misfit with equation (1) (i.e. the best fit to the light intensity profile is a pure exponential fit).[2]

(b) According to Grace, $fracdeV$ is a good indicator to use to classify a galaxy as elliptical or spiral. In her words:

**"The smaller the value of $fracdeV$ - that is, the closer it is to 0 - the more likely it is to be a spiral galaxy."**

Using the information above, explain why this **might possibly** be true. **[1]**

**Solution:**

As mentioned in the background information, $fracdeV = 1$ corresponds to a de Vaucouleurs fit, which best describes the intensity profile of elliptical galaxies.

As $fracdeV$ decreases towards 0, an exponential decay becomes the more accurate. Recall that exponential decay corresponds to the intensity profile of disc-like galaxies. Since spiral galaxies are disc-dominated rather than having a large central bulge, it follows an exponential decay model (equation (2)). Hence, a lower $fracdeV$ value would possibly correspond to a spiral galaxy.

### Luminosity and Brightness

For ease of comparison, we rewrite equation (1) into a form similar to equation (2), as follows:

$$I(r) = I_e \exp\left(-b\left(\left(\frac{r}{r_e}\right)^{\frac{1}{4}} - 1\right)\right), \tag{3}$$

where $b$ is a constant to be determined.

[2]More precisely, the light intensity profile is best approximated by a linear combination of the two equations. $fracdeV$ is the fraction of this best fit linear combination contributed to by equation (1).

The numerical value of the surface brightness $\mu$ of an extended astronomical object is defined as

$$\mu(d) = m - 2.5\log_{10} A + 5.0\log_{10} d, \tag{4}$$

where $A$ is the total surface area of the object, $m$ is the integrated apparent magnitude of the object, and $d$ is the distance between the galaxy and us (Earth).

Here, integrated apparent magnitude is obtained via comparing the combined luminosity of all components of the extended astronomical object with point sources on the magnitude scale. **This means that the usual formulas for apparent magnitudes can be applied in this case**.

(c) There are two different distances, namely, $r$ and $d$. Recall that

- $r$ is the distance between the centre of the galaxy to a point located in the vicinity of the galaxy, and
- $d$ is the distance between the galaxy and us (Earth).

A fellow astronomer is confused with the subtle differences in these two distances, and claims that an equivalent form of equation (4) holds with $d$ replaced by $r$. That is, this astronomer claims that

$$\mu(r) = m - 2.5\log_{10} A + 5.0\log_{10} r.$$

Explain why this equation does not make physical sense. **[1]**

**Solution:**

If it is indeed $r$, since $A$ (total surface area of the astronomical object) does not depend on $r$, then *all* astronomical objects follow a 'decay law' in accordance with the equation of surface brightness. This contradicts the fact that different astronomical objects can have different profiles of surface brightness/intensity against $r$.

Note: Any other convincing arguments are accepted.

(Friendly notification: The following two parts might not be for the faint-hearted!)

(d) With the help of equation (4) and relevant formulas from the Formula Book, show that equation (4) implies that

$$L = k \exp\left(-\frac{\ln 10}{2.5}\left(\mu - 5\log_{10}\frac{d}{10} + 2.5\log_{10} A - 5\log_{10} d\right)\right),$$

where all distances are measured in parsecs, $L$ is the luminosity of the galaxy, $M$ is the absolute magnitude of the galaxy, $\mu$ is the surface brightness of the galaxy, and $k$ is a constant.

(Hint: Two formulas from the Formula Book are relevant. One relates absolute magnitude and apparent magnitude, and the other relates absolute magnitude and luminosity.) **[2]**

**Solution:**

We note that from the equation representing relationship between luminosity and absolute magnitude, that from

$$\frac{L_1}{L_2} = 10^{\frac{M_1 - M_2}{2.5}},$$

we get

$$L = k \times 10^{-\frac{M}{2.5}}.$$

This can be rewritten as a power of $e$ as

$$L = k \exp\left(-\frac{\ln 10}{2.5} M\right). \tag{5}$$

Next, we have the distance modulus formula (in parsecs)

$$m - M = 5\log_{10}\frac{d}{10}.$$

Substituting $m$ in equation (4), we get

$$\mu = M + 5\log_{10}\frac{d}{10} - 2.5\log_{10} A + 5.0\log_{10} d.$$

Rearranging to make $M$ the subject, we have

$$M = \mu - 5\log_{10}\frac{d}{10} + 2.5\log_{10} A - 5.0\log_{10} d. \tag{6}$$

Finally, substituting equation (6) into equation (5), we have

$$L = k \exp\left(-\frac{\ln 10}{2.5}\left(\mu - 5\log_{10}\frac{d}{10} + 2.5\log_{10} A - 5\log_{10} d\right)\right),$$

as needed.

(e) **(BONUS)** Show that equation (3) follows from equation (1) under certain physical considerations. Consequently, show that $b \approx 7.67$. **[2]**

**Solution:**

Recall equation (1):

$$\mu_e + 8.3268\left(\left(\frac{r}{r_e}\right)^{\frac{1}{4}} - 1\right).$$

Substituting into the expression for $L$ in the above part, we obtain an equation of the form

$$L = K\exp\left(-\frac{\ln 10}{2.5}\left(8.3268\left(\left(\frac{r}{r_e}\right)^{\frac{1}{4}}-1\right)\right)\right),$$

where we have collapsed $k$ and the exponential terms in $A$, $d$, and $\mu_e$ into the function $K$. Note that $K$ changes with $A$.

By definition of intensity, $I = \frac{dL}{dA}$. Hence,

$$I = \frac{dL}{dA} = K'\exp\left(-\frac{\ln 10}{2.5}\left(8.3268\left(\left(\frac{r}{r_e}\right)^{\frac{1}{4}}-1\right)\right)\right).$$

Since $I(r_e) = I_e$, it follows that $k'' = I_e$. Hence $b = -8.3268\frac{\ln 10}{2.5} \approx 7.67$.

## Part III Data Analysis on Galaxy Morphology


Next, our computational expert is here to guide us through the process of analysing the data collected by Grace!

Let us recall the objective of this project:

> According to this colour-morphology relation, spiral galaxies are statistically bluer, disk-dominated, and hold more star formation than their elliptical counterparts.

### The Method of Regression

To aid us in analysing this relationship, we will make use of a statistical model known as *logistic regression*[3]. Using such a statistical model, given a particular spiral galaxy's $fracdeV$, we can infer a probability that the galaxy is red (or blue). In this part of the question, we will investigate the correlation between fracdeV of a spiral galaxy with the colour of the spiral galaxy itself. Recall that in general, spiral galaxies have a small $fracdeV$ value. A larger $fracdeV$ value corresponds to one whereby the bulge of the spiral galaxy takes up a larger proportion of the galaxy itself.

A simplistic way to attempt a regression by plot is to use an indicator variable specifying if a spiral galaxy is red or blue. For instance, we could set

$$Y = \begin{cases} 1, & \text{if the galaxy is red, and} \\ 0, & \text{if the galaxy is blue.} \end{cases}$$

We can find the best fit line for $Y$ against the $fracdeV$ of the galaxy via a simple linear model $Y = m(fracdeV) + c$. However, while we can certainly find $m$ and $c$, this method does not make sense! There will be values of $fracdeV$ which are physical but $Y$ is not. For instance, what if $Y = 0.5$? Is that galaxy red or blue? What if $Y = -1$? What if $Y = 2$?

In logistic regression, we relate the 'logarithmic probability' of a galaxy being red with the galaxy's $fracdeV$. More precisely, we model

$$\ln\frac{p}{1-p} = m(fracdeV) + c, \tag{7}$$

where $p$ is the probability that the spiral galaxy is red.

Even so, this poses a problem. How do we assign different values of probability $p$ for each $fracdeV$ value to determine values of $m$ and $c$?

[3]Regression refers to the process of estimating the relationship between a dependent and an independent variable. For example, plotting points on a graph and obtaining the best fit straight line is a regression technique, as it is a process which finds the line best describing the (linear) relationship between the dependent and the independent variable. Logistic regression is similar to this process, but with a curve instead of a straight line.

(f) Si Chen has an idea. For each datum, he wants to assign $p = 1$ for red spiral galaxies, and $p = 0$ for blue spiral galaxies. He will then proceed to obtain a best fit line as in equation (7) to obtain the coefficients $m$ and $c$. Then for each $fracdeV$ value, the probability $p$ such that the galaxy is red can be found from this line.

Explain why this will not work. **[1]**

**Solution:**

$\ln \frac{p}{1-p}$ is not defined for $p = 0, 1$.

### Graphing the Graph

While $p$ is indeed determined through some method using the indicator variable $Y$, to surmount technical difficulties[4], magic happens and you shall assume that we have done the logistic regression for you![5] The logistic regression is done by feeding in a subsample from the citizen science project, Galaxy Zoo, with a sample size of 5433.[6] The independent variable is $fracdeV$. We have found that

$$\ln \frac{p}{1-p} = 8.2(fracdeV) - 4.9. \quad (8)$$

The following table shows a subset of the subsample that was used.

| **Galaxy #** | **Indicator Variable (Y)** | $fracdeV$ |
|---|---|---|
| 1 | 1 | 0.42 |
| 2 | 1 | 0.32 |
| 3 | 1 | 0.20 |
| 4 | 1 | 0.06 |
| 5 | 0 | 0.09 |
| 6 | 0 | 0.32 |
| 7 | 0 | 0.04 |
| 8 | 0 | 0.17 |

**Table 1:** Subset of the subsample used to obtain the coefficients in our logistic regression.

(g) Using the coefficients of $m$ and $c$ obtained in equation (8), explain why equation (8) makes physical sense in the context of spiral galaxies.

(In other words, how do you know that these coefficients are consistent with our background knowledge in galaxy morphology?) **[1]**

**Solution:**

Spiral galaxies are statistically bluer according to the claim of the project. The same is reinforced in the first two parts of this DRQ.

[4] At this juncture, we will ignore this issue of assigning $p$ to avoid other unnecessarily complicated statistical terms. The basic idea is that using a set of data points $(X, Y)$, we are able to obtain the coefficients in the logistic regression equation (7) with the use of certain numerical techniques under certain statistical considerations.

[5] Yay!

[6] Masters, K. L. et al. (2010). "Galaxy Zoo: passive red spirals." Mon. Not. Roy. Astronom. Soc. 405, 783-799. DOI: 10.1111/j.1365-2966.2010.16503.x. arXiv: 0910.4113.

For small values of $fracdeV$ which might correspond to spiral galaxies, say $fracdeV < 0.5$, the probability $p$ of the galaxy being **red** can be calculated by the inequality

$$\ln \frac{p}{1-p} = 8.2(fracdeV) - 4.9 < -0.8,$$

i.e.

$$\frac{p}{1-p} < e^{-0.8}.$$

Solving, we get $p < 0.310$.

This implies that the probability of the galaxy being **blue** is $> 0.690$, i.e. there is a high probability that the spiral galaxy is blue.

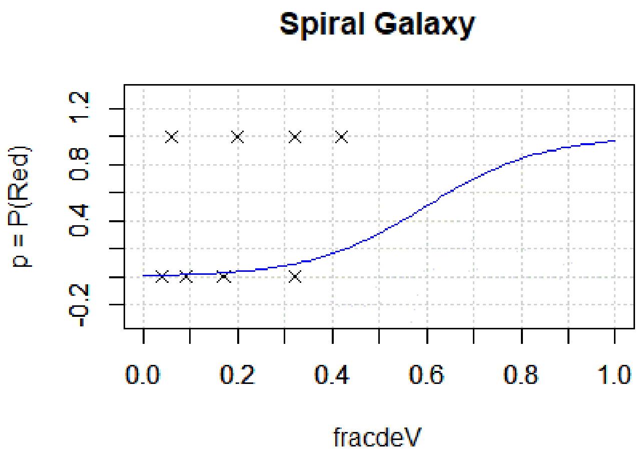
(h) On the same piece of graph paper, using $p$ as the $y$-axis and $fracdeV$ as the $x$-axis, do the following.

(i) Draw the logistic regression curve obtained in equation (8).

(Hint: What is the best way to *draw* a curve, given its equation?) **[3]**

(ii) Plot all the points in the subset shown in Table 1. For each datum, we set $p = 1$ if the spiral galaxy is red, and $p = 0$ if the galaxy is blue. **[1]**

**Solution:**



***Spiral Galaxy Classifier*™**

With our logistic regression complete, we can predict if a spiral galaxy is red or blue given its $fracdeV$. To this end, using equation (8), Si Chen has programmed a spiral galaxy classifier, named *Spiral Galaxy Classifier*™, which only has two possible outputs: "Galaxy is Red", and "Galaxy is Blue". The classifier is programmed such that given a particular $fracdeV$ value, the corresponding probability $p$ is determined from equation (8). It then outputs "Galaxy is Red" with probability $p$, and "Galaxy is Blue" with probability $1 - p$.

(j) Suppose you are provided with the following image.

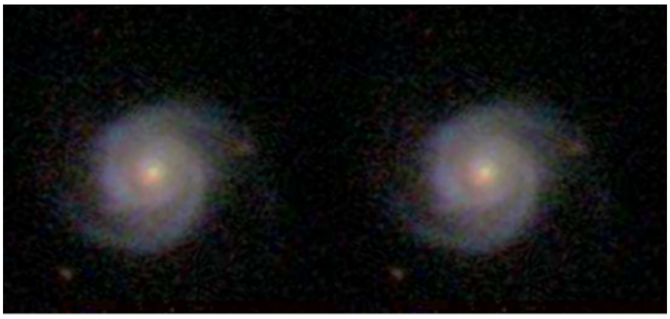


**Figure 3:** A pair of blue spiral galaxies.

You are also told that both spiral galaxies in the image above have $fracdeV = 0.3$. What is the probability that the *Spiral Galaxy Classifier*™ classifies **both** galaxies correctly? **[2]**

**Solution:**

By either reading off the graph or computing from equation (8), we get $p = 0.080$. The probability that the classifier outputs blue is 0.920 on each try. Hence, the probability of classifying both galaxies as blue is $0.920^2 \approx 0.846$.

## Part IV Results and Discussion

As with every project, all the team members must (finally) come together to discuss about interesting outliers and insights that result. Thus, here they are below, back for the last portion of this project!


(k) From the Hubble sequence for galaxies, there is also another type of galaxy which is neither spiral nor elliptical. They are consequently classified as irregular galaxies.

Suppose that we try to classify the following galaxy using the *Spiral Galaxy Classifier*™.

**Figure 4:** Image of a well-known irregular galaxy, the Small Magellanic Cloud.

Is the probability of getting a "Galaxy is Red" result greater than 0.5, or is it less than 0.5? Justify your answer. **[2]**

**Solution:**

$p < 0.5$.

This is because for the irregular galaxy given, the intensity profile as observed from the diagram clearly does not follow the de Vaucouleurs fit (it has a notable absence of a central bulge). Hence, its $fracdeV$ value is low, resulting in a low value of $p$.

(m) From Table 1, we have observed a red spiral galaxy. Discuss and explain how some of these red spiral galaxies could have formed.

(Note: To score full credit for this question, you will need to give at least two possible explanations as to how these red spiral galaxies are formed.) **[3]**

**Solution:**

In this part of the question, we provide some of the theories from the cited paper, "Galaxy Zoo, Passive Red Spirals".

For the astute AC participants, you will realise that the aim of this entire project hinges on the fact different galaxies come in different shapes and sizes. This might lead you to think of other possible types of galaxies that are not mentioned in this entire question, barred galaxies and its hybrids.

- Red spiral galaxies could be barred (i.e. red barred spirals).

  Possible explanation using this fact: Gas required for star formation is redistributed by the presence of the bar inwards towards the core of the galaxy. This means that it is possible that star formation has terminated on the spiral arms of the red barred spiral galaxies. With no new stars forming, the stars that are bluer and more massive with a smaller lifetime will thus die off quickly, resulting in a red barred spiral galaxy.

  (Here, these red spirals thus represent the end stages of spiral evolution)

- Red spiral galaxies are just old spiral galaxies.

  Possible explanation using this fact: If there is star formation, we can always obtain these massive blue stars which are responsible for the blue colour in spiral galaxies.

  If star formation has thus terminated due to the age of the galaxy, then no new stars would be formed. Along the same line of argument as for the first point, we would thus expect these spiral galaxies to be old. One such possible reason for the termination of star formation would be purely based on the fact that it is old, and thus have used up all the gas that is has for star formation.

Note: Any other plausible reasons are accepted, as long as the solution ultimately links back to the termination of star formation.

# Question 4 The Comet Hunt

Comet 67P, also known as Churyumov-Gerasimenko, was first discovered in 1969 with a perihelion distance of 1.29 AU, an aphelion distance of 5.88 AU, and an orbital period of about 6.44 years.

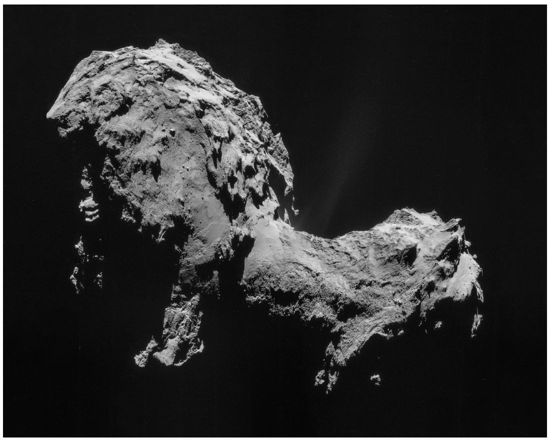


**Figure 5:** Comet 67P/Churyumov-Gerasimenko. Image taken by the *Rosetta* spacecraft.

In the late 1990s, it attracted significant attention in astronomy circles when it was chosen by the European Space Agency (ESA) as one of the objects of interest in their future comet chasing mission. In March 2004, ESA launched the *Rosetta* mission, aimed at obtaining scientific data from comets such as Comet 67P.

The Ariane 5G+ rocket brought the *Rosetta* mission spacecraft to low Earth orbit (LEO). The spacecraft was then further boosted on its path towards comet 67P. It successfully reached Comet 67P in August 2014 and sent back a significant volume of important scientific data over two years.

In this question, we will explore this journey to collect scientific data from Comet 67P.

## Part I Rocketry

Ariane 5G+ is a two-stage rocket, consisting of the core stage (where the payload is contained) and the second stage. The parameters for the core stage and the second stage are given as follows:

| Parameter | Core Stage | Second Stage |
|---|---|---|
| Length | 23.8 m | 3.4 m |
| Diameter | 5.4 m | 5.4 m |
| Empty mass (without fuel and payload) | 12200 kg | 1200 kg |
| Gross mass (fully fuelled without payload) | 170500 kg | 11200 kg |
| Engines | Vulcain1 | Aestus |
| Thrust | 1015 kN | 27 kN |
| Burn time | 605 s | 1170 s |
| Fuel | Liquid hydrogen | Monomethylhydrazine |

**Table 2:** Data for the two stages of Ariane 5G+.

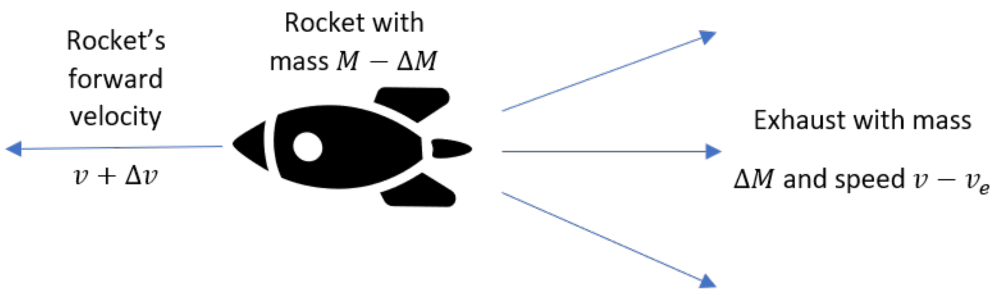
**Figure 6:** A fun picture of Ariane 5G+.

(a) A rocket of mass $M$ burns a fuel mass of $\Delta M$ in a small amount of time $\Delta t$. It then ejects the burnt fuel with velocity $v_e$ and changes the rocket's velocity by $\Delta v$.

Draw a simple diagram illustrating this situation.

(Note: A single, well-labelled diagram with all relevant details will suffice. It is not necessary for $\Delta t$ to appear in your diagram.) **[2]**

**Solution:**




(b) By respectively using Newton's second and third laws of motion, or otherwise, state two equations relating the quantities in the diagram drawn in part a.

(Note: In considering the second law, you should be using the form $F = \frac{dp}{dt}$. In place of Newton's third law, you may alternatively use conservation of momentum.) **[2]**

**Solution:**

From Newton's second law,

$$\frac{\Delta M}{\Delta t} v_e = F_{\text{thrust}}.$$

From Newton's third law (or conservation of momentum),

$$\Delta M(v - v_e) = (M - \Delta M)(v + \Delta v) - Mv, \Delta M v_e \qquad = M\Delta v$$

For the second equation, both the simplified version and the first order approximation (i.e. ignoring the term involving $\Delta M \Delta v$) are accepted.

(c) Using the thrust given and any other relevant data, calculate the exhaust velocity $v_e$ of the fuel used for both stages. You may assume the exhaust velocity remains constant as the rocket accelerates. Ignore any atmospheric effect. You may also assume that the burn rate of the fuel is constant over the entire period of the burn.

(Note: You may find the equation(s) you have written in part **b** useful for this calculation. Remember also that $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.) **[2]**

**Solution:**

Since the burn rate of the fuel is constant, for the core stage we have

$$\frac{\Delta M}{\Delta t} = \frac{170500 - 12200}{605} \approx 261.65\,\mathrm{kg\,s^{-1}}\,.$$

For the second stage we have

$$\frac{\Delta M}{\Delta t} = \frac{11200 - 1200}{1170} \approx 8.54\,\mathrm{kg\,s^{-1}}\,.$$

Using the second law's equation derived above, together with the provided values for thrust, we deduce the exhaust velocities are $\frac{1015}{261.65} \approx 3.88\,\mathrm{km\,s^{-1}}$ for the core stage, and $\frac{27}{8.54} \approx 3.16\,\mathrm{km\,s^{-1}}$ for the second stage.

(d) Calculate the speed of the rocket after the burning of each stage of the rocket, assuming the rocket travels vertically upwards. Take the payload weight to be the weight of the *Rosetta* mission satellite at 3000 kg.

(Note: Remember that the second stage burns first, so at the end of burning the second stage, you should be left with an empty second stage and a fully loaded core stage. The second stage then detaches and the core stage starts to burn.) **[2]**

**Solution:**

Using the ideal rocket equation from the Formula Booklet, for the second stage we have

$$\Delta u = 3.16 \ln \frac{3000 + 170500 + 11200}{3000 + 170500 + 1200} \approx 0.176\,\mathrm{km\,s^{-1}}\,.$$

For the core stage,

$$\Delta u = 3.88 \ln \frac{3000 + 170500}{3000 + 12200} \approx 9.447\,\mathrm{km\,s^{-1}}\,.$$

Hence the speed at the end of second stage is $0.176\,\mathrm{km\,s^{-1}}$, and the speed at the end of first stage is $0.176 + 9.447 \approx 9.623\,\mathrm{km\,s^{-1}}$.

(e) The rockets are launched due east from the Guiana Space Centre, located in French Guiana, South America, with a latitude of 5.1673° N, rather than in Europe where most of the parts are manufactured. Suggest a(n) astronomical/physical reason for this choice of launch location.

(Note: Practical reasons such as cheaper labour cost and better weather are **NOT** accepted.) **[1]**

**Solution:**

It reduces the $\Delta v$ required to bring the rocket into space, as part of the velocity can be provided by the earth's rotation.

(f) Contrary to some spacecraft operating entirely in space (e.g. Apollo programme's Lunar Module, Soviet Luna program), rockets (like Ariane 5G+) that take off from Earth are multistage rockets.

(i) Suggest a reason why a multistage design is used for rockets taking off from Earth, rather than a single stage design. Support your answer with relevant calculations. **[2]**

**Solution:**

With a single stage, it is necessary to carry all machinery required to bring the fuel to a higher altitude before it can be discarded. This wastes some of the kinetic energy generated by the fuel burning as a larger mass needs to be propelled.

Assuming Ariane 5G+ is instead a single stage rocket with the combined gross and empty masses of both the core and second stages. We can estimate the final velocity of the rocket by assuming the exhaust velocity is equal to that of the second stage velocity.[7]

The final velocity of the rocket will be

$$\Delta u = 3.16 \ln \frac{3000 + 170500 + 11200}{3000 + 12200} \approx 7.892\,\mathrm{km\,s^{-1}}\,.$$

This is significantly lower than the final velocity of the two-stage calculation, i.e. the fuel usage is less efficient.

(ii) **(BONUS)** Suggest another reason why this is so. Calculations are NOT required in this part. **[1]**

**Solution:**

The nozzle is the most efficient when the pressure of the exhaust at the end of the nozzle is approximately equal to the pressure of the surrounding atmosphere. Any discrepancy between the two pressure will result in turbulence and thus reduce the efficiency of the nozzle.

Since the atmospheric pressure varies with altitude, any rocket nozzle can only be designed for maximum efficiency at a specific altitude. Thus, a single stage rocket's nozzle will be experiencing more time in inefficient use, while nozzles on multistage rockets can be calibrated to the different altitude ranges they are expected to work in. This results in more chemical energy converted into kinetic energy of the payload (and not lost to turbulence), and thus less fuel is required.

For most low Earth orbits, much of the burning takes place within the Earth's atmosphere, rather than in the near vacuum of space. Thus the nozzle design is more similar to that of the second stage of the Ariane 5G+ rocket, as it is designed for use with atmosphere. In fact, the answer we obtain for single stage rocket in the previous part should be a upper estimate of the final velocity, due to efficiency issue highlighted earlier.

Note: Full credit is awarded for any mentioning of rocket's nozzle efficiency as the reason. Other sensible and logical alternative reasons are also accepted.

## Part II Orbital Mechanics

Up to 1840, Comet 67P was in orbit around the Sun with a perihelion distance of 4 AU. A close encounter with Jupiter altered its orbit to one with a perihelion distance of 3 AU.

Over the next century, further encounters reduced its perihelion distance to 2.77 AU. Finally, in 1959, another close encounter with Jupiter changed its orbit to the one it currently has today, with a perihelion distance of 1.24 AU.

This proximity to Earth's orbit makes it an excellent target for scientific studies. The table below contains astrometric data of Comet 67P, obtained in 2014 as part of the *Rosetta* mission.

| Nucleus overall dimensions | 4.34 km ×2.60 km ×2.12 km |
|---|---|
| Small lobe dimensions | 2.50 km ×2.14 km ×1.64 km |
| Large lobe dimensions | 4.10 km ×3.52 km ×1.63 km |
| Mass | $1.0 \times 10^{13}$ kg |
| Density | 533 $kg/m^3$ |
| Orbital period | 6.45 years |
| Perihelion distance from Sun | 186 million km (1.243 AU) |
| Aphelion distance from Sun | 849.7 million km (5.68 AU) |
| Orbital eccentricity | 0.640 |
| Orbital inclination | 7.04° |
| Surface temperature | −93°C to 53°C |

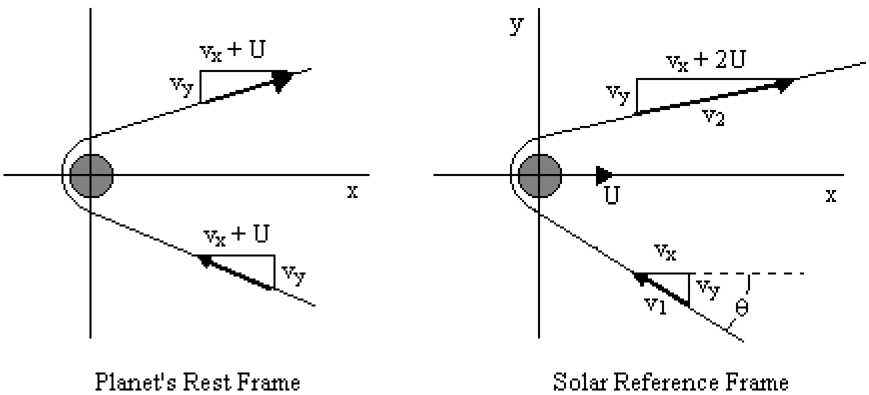
**Table 3:** Astrometric data of Comet 67P/Churyumov-Gerasimenko.

You are also given that its current albedo is 0.06, and that Comet 67P is likely to be a result of a fusion of two smaller comets.

(g) As mentioned above, in 1959 the orbit of Comet 67P was altered due to its close encounter with Jupiter. With a diagram, suggest a reason for and briefly explain this alteration. **[2]**

**Solution:**

It likely underwent a gravitational slingshot effect via Jupiter, significantly boosting its orbiting velocity and thus also significantly reducing its orbital semimajor axis.




(h) The flight path of the *Rosetta* mission spacecraft was controlled by its onboard computer, which is notably much less powerful than those at the ESA Mission Control. Suggest why the onboard computer is used, and **not** the computers at the ESA. **[1]**

**Solution:**

Since the semimajor axis of the comet is currently about 1.243 AU, and from relativity we know what nothing can travel faster than the speed of light, this means that information from Earth takes a minimum of $0.243 \times 8.3 \approx 2.02$ min to reach the spacecraft.

This delay is too large for the purposes of flight control, where rapid responses are often needed.

(j) For this question, assume that Comet 67P is a uniform sphere. The Hill radius $r_H$ of Comet 67P denotes the distance from Comet 67P that a satellite, such as the *Rosetta* mission spacecraft, would transit from orbiting the Sun to orbiting the comet. The Hill radius is given by the formula

$$r_H = r\sqrt[3]{\frac{m}{3M}},$$

where $r$ is the distance between Comet 67P and the Sun, $m$ is the mass of Comet 67P, and $M$ is the mass of the Sun.

Compute the average Hill radius of Comet 67P. **[2]**

**Solution:**

Using the average value of the distances between the comet and the Sun,

$$r_H = \frac{1.243 + 5.68}{2} \times 1.496 \times 10^8 \times \sqrt[3]{\frac{1 \times 10^{13}}{3 \times 2 \times 10^{30}}} \approx 614\,\text{km}\,.$$

(k) In reality, the *Rosetta* mission spacecraft does *not* orbit around Comet 67P at the distance calculated in part j. What is the **most** important reason why? **[1]**

**Solution:**

The Hill radius depends on the distance between the comet and the Sun. This varies with the comet's motion around the Sun, thus for about half of the comet's orbit, sending a satellite to orbit Comet 67P at the distance calculated above will result in an heliocentric orbit instead.

Note: DO NOT ACCEPT any answer arguing from the non-spherical nature of the comet. At the orbital distance calculated above, the effect of the non-spherical nature of the comet is not as significant as the changing orbital distance.

(m) Suggest a possible reason why comets such as Comet 67P are of interest to astronomers. **[2]**

**Solution:**

Any one of the following reasons (list is not exhaustive):

- Test the technology required to send a probe to a comet (it has not been done before).
- Understanding the composition of materials during the formation of the solar system, and thus constructing better models for its formation.
- Understanding how two smaller comets could fuse together to form objects such as Comet 67P (as that is its suspected formation mechanism, and no comet with a similar formation mechanism has been studied before).
- Understanding the effect of dust and comet ejecta on the trajectory of comets, and its effects on the surface and subsurface of the comet.

Note: Any other plausible and sensible answers are accepted.

Read the following excerpt from NASA regarding air quality classifications.

*'Air quality classifications, as defined by FED-STD-209E, are specified by the maximum allowable number of particles per cubic foot (or cubic meter) of air. The name of the class in English units (the usual convention in the U.S.) is taken from the maximum allowable number of particles,* 0.5 μm *and larger, per cubic foot. Class* 350, 000 *air is typically referred to as a "good housekeeping area" and is suitable for most integration and assembly operations. Class* 00, 000 – *Class* 1, 000 *air is referred to as a "cleanroom" and is required for installation of most space system hardware.'*

Next, consider the flight path of the *Rosetta* mission spacecraft.

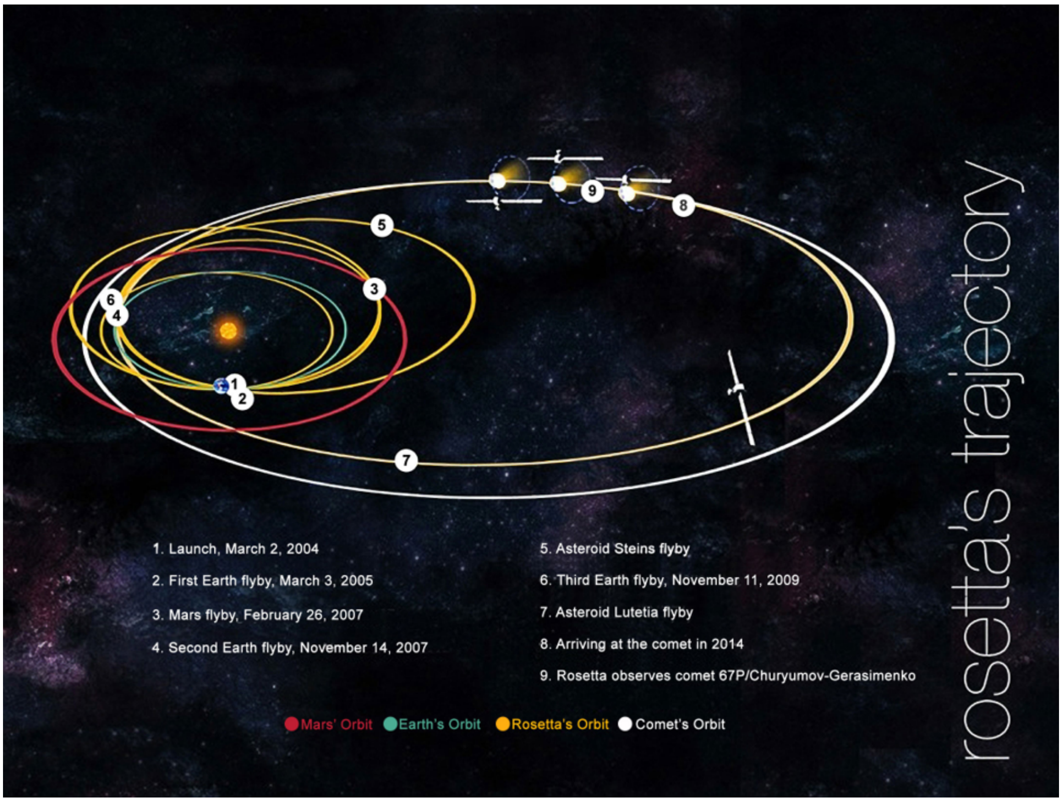



**Figure 7:** *Rosetta*'s mission flight path.

(n) Most spacecraft parts are manufactured under cleanroom conditions. The parts for the *Rosetta* mission are no exception. By referring to the flight path of the *Rosetta* spacecraft to Comet 67P in Figure 7, explain why this is necessary in the particular case of the *Rosetta* mission spacecraft. **[1]**

**Solution:**

The *Rosetta* mission has a chance of colliding with Mars on its close approach during its gravitational assist. Thus it is necessary to prevent Earth's bacteria form contaminating the surface of Mars in the event of a crash on the surface of Mars.

Note: DO NOT ACCEPT any answer about preventing contamination on the surface of Comet 67P. It is not a requirement to prevent biological contamination on comets.