AstroChallenge 2021
Senior Team Round

Monday 7th June 2021

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **19** printed pages, including this cover page.
2. You are required to keep your microphone and camera on at all times throughout the round.
3. You are not allowed to use your keyboard at all times, but you may use your mouse to scroll through the question paper as well as switch to the formula booklet.
4. Any materials other than the Question Paper, Formula Booklet, and **ONE** A4-sized cheat sheet held by **ONE** team member only, are strictly prohibited.
5. You have **2 hours** to attempt all questions in this paper.
6. Write your answers on blank pieces of A4 paper or graph paper. Do **NOT** mix solutions for different questions on the same sheet of paper.
7. You will be given time after the paper to collate your answers. You should collate your answers into **separate PDF files** for each question.
8. It is *your* responsibility to ensure that your answer scripts have been submitted.
9. The marks for each question are given in brackets in the right margin, like such: [**2**].
10. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

Question 1 Short Answer Questions

Part I Stellar Systems

Consider the following star.

Name	X Astrochallengae
Right Ascension	2h 30min 49s
Declination	$67°28'43''$
Distance	142ly
Apparent Magnitude	4.5
Absolute Magnitude	1.31

Table 1: Information regarding X Astrochallengae.

(a) I currently see this star system at the zenith. Using this information, if able, derive my latitude, longitude as well as the local sidereal time. If you are unable to, you should explain why. **[3]**

(b) Without using Table 1, suggest a method to determine the distance to this star system. Explain why your suggested system is appropriate. **[3]**

(c) You decide to go to Lake Tekapo in New Zealand (43°53′S 170°31′E), a place with very dark skies. However, you find that you never be able to see X Astrochallengae from New Zealand even with a telescope. Suggest a reason why. **[1]**

(d) Suppose we want to determine if the X Astrochallengae system contains planets. Suggest a method and briefly explain how it works. **[3]**

Part II Dubious Statements

This part comprises 5 statements. For each statement, indicate clearly whether it is **TRUE** or **FALSE**.

Support your answer with no more than 6 sentences, including any assumptions where required. You may draw up to one additional diagram if they aid your explanation.

Mathematical working is not required, and there are no errors in any of the statements below.

Each statement is worth 2 marks, **attributed only to the quality of the justification**.

(e) It is only possible to observe at most two eclipses (either lunar or solar) during a single eclipse season. **[2]**

(f) The sky is blue because it reflects the colour of the ocean. **[2]**

(g) It is possible to observe Venus with the naked eye at any time of the day and night. **[2]**

(h) A white dwarf star that has a mass of 1.5 solar masses ($1.5 M_\odot$) is considered stable and will eventually turn into a black dwarf. **[2]**

(j) Earth is the only planet in the solar system that experiences aurorae. **[2]**

Question 2 Operation: BINARY

Project: NEUTRON

You are an astronaut working for the United Nations Space Command (UNSC) in the far future. You are currently onboard your ship, the *Pillar of Autumn*, a few AUs away from a binary neutron star system. Your task: analyse the binary star system.

(a) What are neutron stars mainly composed of? **[1]**

Your ship is stationary and in the plane of the orbit of the binary neutron stars. You go to cryo-sleep to allow your ship AI to collect as much data as possible from the binary star system, and once sufficient data has been collected, you are awakened by your ship. Shown below is the graph of the brightness of the binary system against time obtained by your ship.

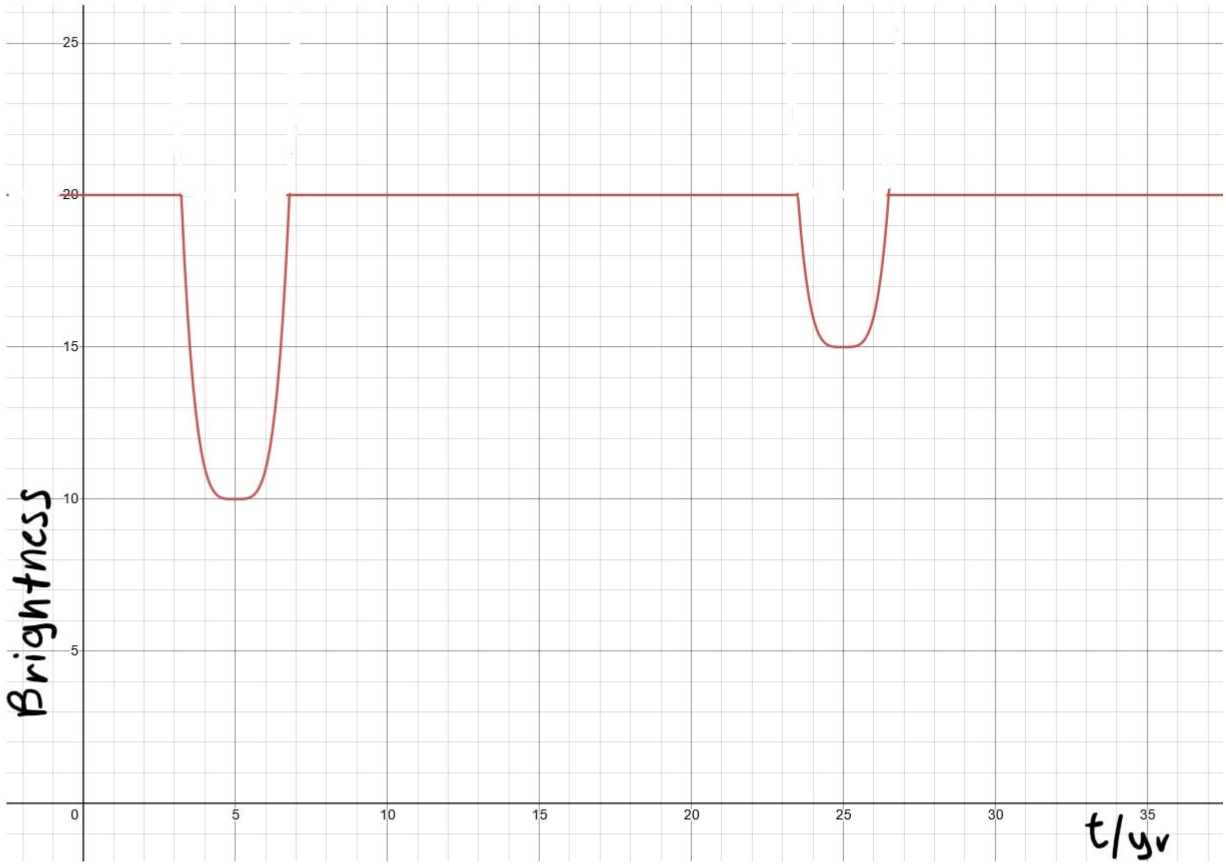


Figure 1: Graph of brightness of the binary star system against time

(b) What is the period of the binary system? **[1]**

(c) Using Newton's law of gravitation and Newton's second law, show that the period, P, of the binary system is $P = \sqrt{\frac{4\pi^2 R^3}{G(m_1+m_2)}}$. Assume both orbits are circular. **[4]**

(d) Using Part c, show that the sum of the masses of the binary stars is

$$m_1 + m_2 = \left(\frac{P}{2\pi G}\right)(v_1 + v_2)^3.$$

[2]

(e) The radial velocity of the two stars have been determined to be $v_{1r} = 20\text{kms}^{-1}$ and $v_{2r} = 40\text{kms}^{-1}$. Determine the masses of the binary stars. **[4]**

Project: GRAVITY

Your next task for the UNSC: study gravitational waves generated by a binary black hole system. You will analyse gravitational waves using the wave equation.

(f) In your own words, describe what a gravitational wave is. **[1]**

Like all types of waves, a gravitational wave can be analysed with a wave equation. A typical wave equation has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right).$$

Thankfully, the full strength of the general wave equation is not needed. A simplified version of the wave equation has the form

$$y(x,t) = A\cos(kx - \omega t), \tag{1}$$

where y is the vertical displacement of the wave, x is the horizontal displacement of the wave, t is time, A is the peak amplitude, k is the wavenumber given by $\frac{2\pi}{\lambda}$, and ω is the angular frequency given by $2\pi f$ (where f is the frequency). The vertical displacement of the wave is dependent on two variables, time and the horizontal displacement of the wave.

You will use Equation 1 to model a gravitational wave. Such waves travel at the speed of light, $c = 3 \times 10^8 \text{ms}^{-1}$.

Earth is located 1.3×10^9 light years away from the black hole binary system GW150914. Gravitational waves from GW150914 were detected at the newly rebuilt **Laser Interferometer Gravitational-Wave Observatory** (LIGO) located on Earth, a memoir and testament to the historical original.

(g) Take $t = 0$s as the time the gravitational wave was generated and $x = 0$ as the source of the gravitational wave. The gravitational waves are determined to have frequency 150Hz and peak amplitude $A = 2 \times 10^{-18}$m. Determine the amplitude of the wave when it is detected by LIGO. Take a year to be 365 days. **[2]**

(h) Why are gravitational waves difficult to detect on Earth? **[1]**

This futuristic version of LIGO is faithful to the original systems used in LIGO of the early 21^{th} century.

(j) State the method LIGO uses to detect gravitational waves. **[1]**

(k) Explain how the method used in Part j allows LIGO to detect gravitational waves. **[1]**

Question 3 All about Limits

Many often know of the limit after which white dwarfs become unstable. But what actually happens inside the white dwarfs and what leads to this instability? We examine the physics behind degeneracy pressure and how this leads to both the stability and instability of white dwarfs and neutron stars. To do this, we model white dwarfs and neutron stars as Fermi gases.

Part I Fermi Gas

Let us first examine a **Fermi gas** in more detail. A Fermi gas is simply a gas of non-interacting fermions. Fermions are particles that include protons, neutrons, and electrons.

(a) In classical physics, what happens to the momentum and the energy of the fermions in a Fermi gas as the temperature T approaches 0? **[1]**

Fermions obey the **Pauli exclusion principle**, which states that:

No two fermions can occupy the same quantum state.

Let us examine clearly what this means in two different ways. We know from H2 Chemistry that energy levels are discrete and quantised. The Pauli exclusion principle tells us that there is a limited number of 'slots' that can be filled for each energy level. When these 'slots' are filled up, electrons are forced to occupy higher energy levels.

(b) Explain how this leads to the formation of pressure. **[2]**

Let us now examine this from the momentum perspective. We have learnt from H2 Physics about the Heisenberg uncertainty principle, which gives a relation between the uncertainty Δx in a particle's position and the uncertainty Δp in a particle's momentum. The uncertainty principle is expressed by the following inequality.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

When the fermions are compressed into a very tight space, they have a very well-defined position.

(c) By considering the momenta of the fermions as well as the Heisenberg uncertainty principle, briefly explain how there is formation of pressure. **[2]**

Now that we have qualitatively established the mechanism behind the formation of degeneracy pressure due to fermions, we can move on to apply this!

Part II White Dwarfs

In main sequence stars, nuclear processes occurring within the core of the star result in an outwards radiation pressure which balances the inwards gravitational force, giving rise to **hydrostatic equilibrium**. In more compact objects however, the pressure arises due to the degeneracy of fermions (such as protons, neutrons and electrons). Since we are examining white dwarfs, we will specifically be looking at electron degeneracy pressure.

We model the white dwarf as a Fermi gas. In general, we can write the degeneracy pressure due to an **non-relativistic** fermion as

$$P_{\text{fermion}} = knE_F,$$

where k is a constant, E_F is the energy of the fermions, and n is the number density of the fermions, related to the density of the star by the equation

$$n \approx \frac{Z\rho}{Am_p},$$

where Z, A, and m_p are the atomic number, mass number, and mass of the protons respectively. ρ is the density of the Fermi gas. This quantity is important in this discussion.

Given that the momentum of the electrons is given by $p_F = \hbar(3\pi^2 n)^{\frac{1}{3}}$, we can obtain an expression for the electron degeneracy pressure of an electron,

$$P_{\text{electron}} = \frac{k\hbar^2}{m_e}(3\pi^2)^{\frac{2}{3}}\left(\frac{Z}{Am_p}\right)^{\frac{5}{3}}\rho^{\alpha}. \tag{2}$$

(d) Show that $\alpha = \frac{5}{3}$. **[2]**

(e) With reference to Equation 2, explain why, in our discussion, we have considered electron degeneracy pressure but not neutron or proton degeneracy pressure. **[2]**

This outward electron degeneracy pressure balances the inward gravitational pressure. Inward gravitation pressure can be derived using the virial theorem, but we shall spare you the pain of doing so. The equation is

$$p_{\text{grav}} = -\frac{G}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} M^{\frac{2}{3}}\rho^{\frac{4}{3}}.$$

Thus we have relations between the exponents of ρ, and the pressures due to gravity and electron degeneracy.

We are now in a position to look at the stability of the system.

Suppose the white dwarf (or Fermi gas) is at equilibrium and is **non-relativistic**. We give it a small perturbation inwards such that the white dwarf contracts a little. This causes the density of the white dwarf ρ, to increase by a little. This is illustrated in the figure below.

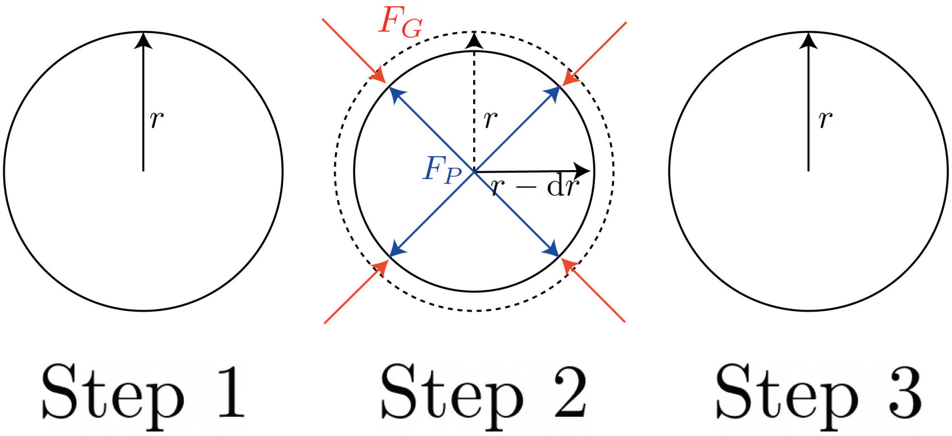


Figure 2: How the equilibrium of the system is restored. **Note that both the arrows representing the forces due to gravity and pressure are there for illustration purposes only and their lengths do not represent the actual magnitudes of the forces.**

(f) This contraction will cause an imbalance in the pressures due to gravity and electron degeneracy. Explain how this imbalance allows the white dwarf to return to its original equilibrium. **[3]**

Now, let us examine the case for a **relativistic** gas. We now have to use the relation $E_F = p_F c$ to obtain the electron degeneracy pressure for a relativistic electron

$$P_{\text{electron}} = K\rho^{\frac{4}{3}},$$

for some constant K.

(g) By doing a similar analysis to what we have done for the non-relativistic gas, explain why there is an absence of stability. **[2]**

(h) **(BONUS)** Notice that for a relatvisitic Fermi gas, the exponent of the density term in the electron degeneracy pressure is lower than that of a non-relativistic gas. This is counter-intuitive, because the electrons in a relativistic Fermi gas are moving faster. Explain why this is the case. **[1]**

We can now estimate the mass above which the electrons in a white dwarf will behave relativistically. This occurs when

$$P_F \geq m_e c,$$

which leads to

$$M \geq \left(\frac{1}{2m_p}\right)^2 \frac{3\sqrt{\pi}}{2} \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.$$

(j) Evaluate this limit in terms of the mass of the Sun, $M_\odot$. What is the name of this limit? **[2]**

Hopefully, this has been able to give you more insight behind why a white dwarf behaves the way it does beyond this limit!

Part III Neutron Stars

Our analysis above for the pressure, energy, and momentum are for a Fermi gas. Naturally, we expect it to hold for neutrons as well. By using a modified version of the number density for neutrons, we are able to obtain a similar expression for a lower limit of the mass above which neutrons in a neutron star will behave relativistically, given by

$$M \geq \left(\frac{3\sqrt{\pi}}{2m_p^2}\right)\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.$$

(k) Evaluate this limit in terms of the mass of the Sun, $M_\odot$. **[2]**

Neutrons have a much shorter de Broglie wavelength than electrons at a given energy, which results in them being spaced much more closely than electrons in a Fermi gas. This means that the pressures within neutron stars are much higher than those of white dwarfs.

(l) Although the actual limit for M is still being improved, it is believed to be around 2 to $3M_\odot$. Suggest a possible explanation for the inaccuracy of the calculated limit in Part k. **[2]**

Question 4 A Certain Academic City

In a certain (fictional) academic city[1], much of the activity taking place is devoted to academic studies and research. You have managed to secure a tour through several astronomy-related facilities! First, meet your guides.

Matou	**Kotomi**	**‘Akuta’**
		
Unluckiest Student	*Denpa Ojou*	*Smartest One*

“Good morning!” Kotomi[2] greets you brightly.

“H-Hi!” Matou goes next, clearly nervous. “Um...um...you’re with me first, and...”

“Get on with it,” Akuta scoffs, supremely disinterested. “You guys wanna do buddy-buddy, leave me outta it. Tch.”

“We need to be polite!” Kotomi scolds him with a long-suffering air. “And Matou-san[3], calm down!”

It isn’t before long that the three start arguing. You wonder if you’ll really be okay with them.

[1] Credits: Scenario is shamelessly inspired by the world of Toaru Majutsu no Index. All humanoid character models were created using the avatar creator CHARAT. Koro-sensei (Ansatsu Kyoushitsu) head was self-drawn. Nurufufufu∼

[2] ‘Denpa Ojou’ translates literally to ‘Electromagnetic Young Lady’. Colloquially, it can also mean ‘Weird Young Lady’. It’s a pun.

[3] ‘-san’ is a polite Japanese honorific. The names are Japanese, because the scenario inspiration is a Japanese light novel/anime series.

Part I A Certain Colourful Index

Your tour starts with Matou's lesson at the city's observatory. Here, you are introduced to his ever-hungry friend. Who for some reason is chomping on his head.

...You wonder if you need the right *filter* to view that particular interaction.

The Colour Index

The *colour index* of an astronomical object is a numerical value that indicates its colour. It is a fairly common occurrence in plots involving colour – for example, most modern Hertzsprung-Russell diagrams plot absolute visual magnitude against the B–V colour index.

To understand how the numerical value is derived, we first need to understand the concept of a filter. A *filter* restricts light observed to a certain range or band of wavelengths, and is named for the wavelengths allowed through. For example, a blue filter allows a band of wavelengths associated with blue light, and *only* this band of wavelengths, through.

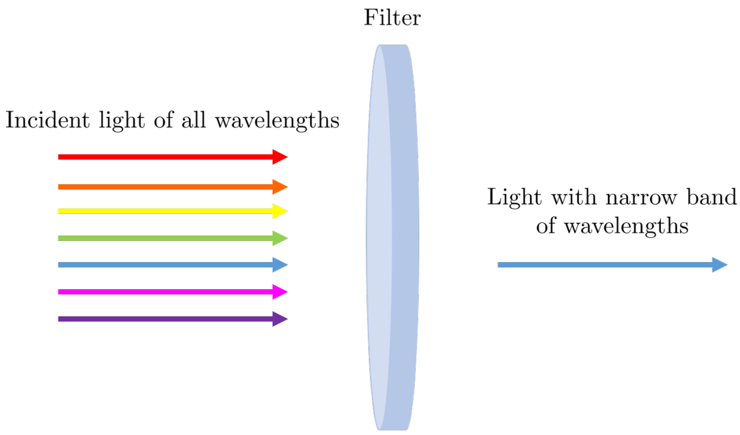


Figure 3: A filter blocks all wavelengths except a narrow band.

In optical astronomy, three filters are the most commonly used – **U**ltraviolet, **B**lue, and **V**isual – forming the UBV system. Sometimes, two additional filters – **R**ed and **I**nfrared – are used, forming the UBVRI system. To determine a colour index, we must specify a choice of two filters to use. For example, if we specified the filters B and V, we obtain the B–V colour index. The B–V colour index of an astronomical object is

$$\text{B–V colour index } = m_{\mathrm{B}} - m_{\mathrm{V}},$$

where m_{B} is the apparent magnitude of the object through the B filter, and m_{V} is the apparent magnitude of the object through the V filter. Analogous computations hold for other colour indices.

(a) True to his unlucky title, Matou is asked the first question.

"Under ideal conditions, knowing **only** the colour index is enough to determine colour, an intrinsic property. And this is because the colour index is a quantity independent of distance. But the colour index is an apparent magnitude difference, and apparent magnitudes depend on distance. So why doesn't the colour index?"

As the star (pun completely intended) visiting student, help him out by answering the question. **[2]**

Even More Index!

Under ideal assumptions, a single colour index suffices to determine colour. However, reality is often disappointing. Various factors (e.g. interstellar extinction) can affect the colour index value, and different colour indices have different sensitivities to various such phenomena. It is for this reason that multiple colour indices are used.

Star	V apparent magnitude	B–V index	V–R index	R–I index
He 767	10.69	0.62	0.25	0.26
He 601	11.43	0.73	0.32	0.27
AP 25	12.25	0.88	0.41	0.34
AP 55	13.91	1.11	0.56	0.41
AP 86	14.31	1.32	0.81	0.60
AP 20	15.66	1.55	1.28	1.07
AP 60	15.82	1.70	1.34	1.13

Table 2: Photometry data of some low-mass stars in the α Persei cluster, approximately 570 ly from Earth. Edited from Stauffer et al. (1985).

Your next task is to analyse this data. Somewhere to your side, Matou starts making a sound like a dying cat.

(b) From analyses of open clusters, it is expected that a colour-colour diagram displays a linear relationship. On a **B–I against B–V graph**, do the following.

(i) Plot the seven stars in Table 2. **[2½]**

(ii) Draw a line of best fit and identify the outlier(s). **[1½]**

You may assume that all stars in Table 2 lie on the main sequence. In general, for main sequence stars, the **V absolute magnitude against B–I plot** is linear for B–I between 0 and 5 (the A0V and M0V spectral classes have B–I index values approximately 0 and 3.6 respectively). Your own **B–I against B–V** graph has deviations from linearity, implying that there is a certain factor that the B–V readings are more sensitive to as compared to the B–I readings.

(c) With the aid of your graph and of Table 2, as well as your own knowledge, deduce a possible candidate for this factor. Explain your reasoning. **[2]**

Part II A Certain Scientific Rail/Gun

Your ~~ordeal~~ experience with Matou has ended. Now, Kotomi and Akuta have joined forces to, in two halves, introduce you to a study of the cosmic ray energy spectrum.

"Hello again!" Kotomi greets you cheerfully, a tag stating *Cosmic Ray Energy Lab, Level 5* dangling from her lanyard. "Since I love electromagnetism, I'll be the one introducing you to shocks and magnetic reconnection. Then Akuta will discuss the energy spectrum, okay?"

A Shocking Topic

Many astrophysical phenomena are associated with shocks. Examples include supernovae and the bow shock at the magnetosphere. A *shock*, or *shock wave*, is defined as a disturbance in a fluid moving faster than the local speed of sound, such as a plane travelling through air at supersonic speeds compressing air in front of it to form a pressure (and shock) front. When passing through a shock, pressure, temperature, and medium density are nearly discontinuous with both sides of the shock.

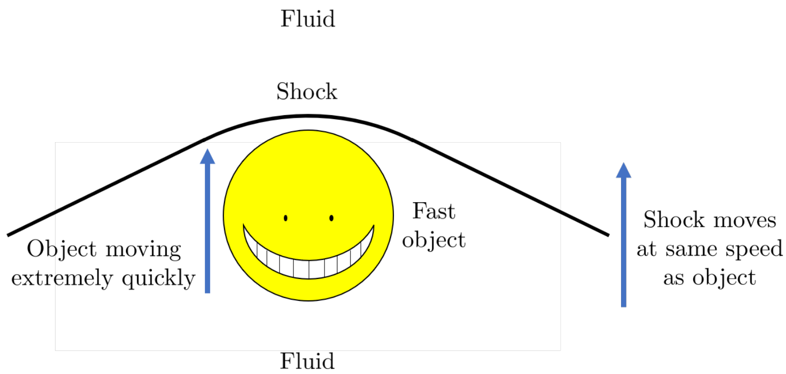


Figure 4: A fast-moving object in a fluid and associated shock.
Note that object needs to move faster than the local speed of sound to create a shock.

In media such as air or water, the primary form of energy transfer stems from collision of particles. In the low density of space, however, most shocks are *collisionless*. That is, shocks form in fluids (primarily plasma) where energy transfer between particles are mediated through means other than collisions. An example of this is the bow shock formed by the interaction of the solar wind with Earth's magnetosphere.

"Okay! A question for you!" Kotomi smiles sweetly.

(d) Briefly explain how energy transfer might occur between charged particles in low-density plasma. **[1]**

"Here is the important picture," Kotomi says. "Because of the discontinuity, we can treat the two sides of the shock as 'walls', or 'mirrors'. A particle trapped between the two will end up bouncing between them. We can think of each collision as an elastic particle-wall collision, but with a small difference."

Figure 4 demonstrates the shock travelling through the fluid from an observer's frame of reference. It is therefore clear that from the shock's frame of reference, particles in the fluid must travel from *upstream*, i.e. in front of the shock where the particles have not yet encountered the shock, to *downstream*, i.e. behind the shock where the particles have encountered the shock.

"The mirrors' *positions* are stationary relative to the shock," Kotomi explains, holding up a finger in a lecturing pose. "But in this frame, the net velocity of the *particles* in each mirror isn't zero. There is a continuous movement of particles from upstream to the downstream, after all. What this means is that a trapped particle will bounce off mirrors that only *look* stationary."

Her hand suddenly slams into the screen, now displaying Figure 5. "But looking is not the same as being! Because the particles in the mirror have non-zero net velocity, any collision with that mirror will be like a collision with a moving wall with that net velocity! Please remember this!"

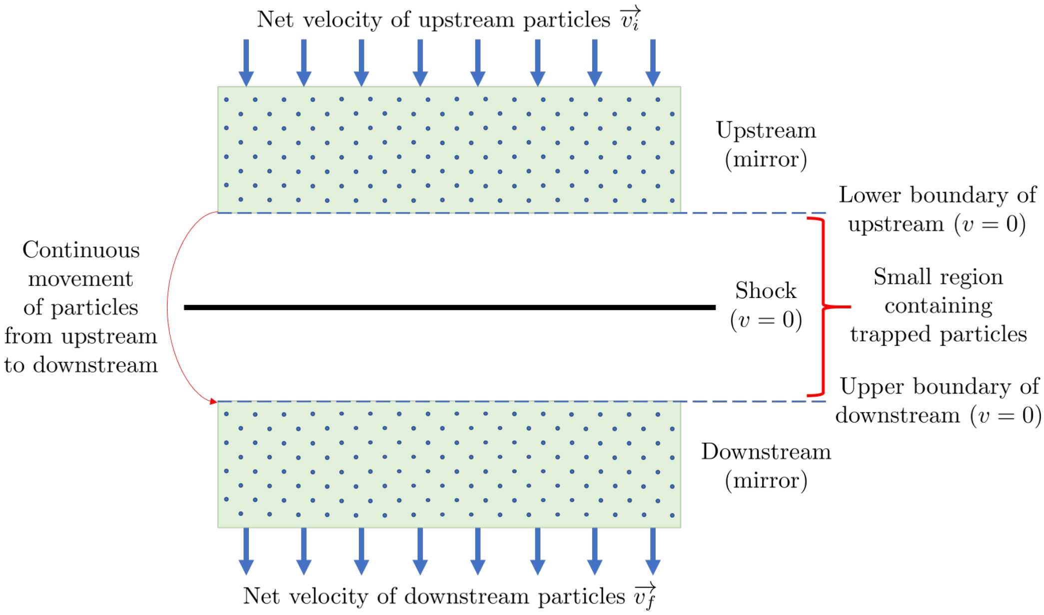


Figure 5: A high-zoom (not to scale) representative picture of the mirrors and the region around the shock. Velocities are taken in the reference frame of the shock.

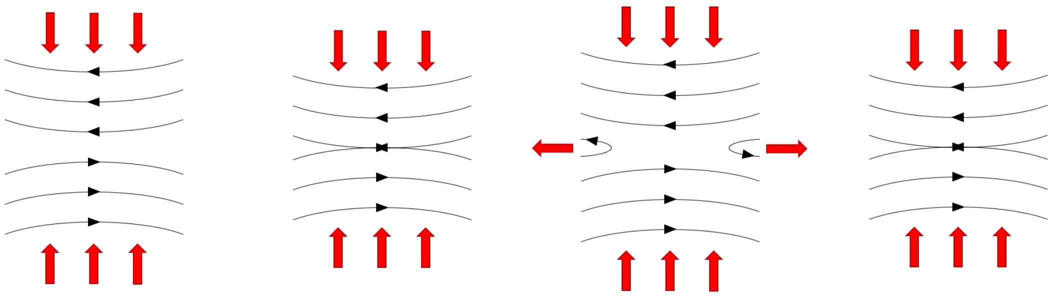
Recall that for an elastic collision, relative speed is always conserved.

(e) Assume that both $\vec{v_i}$ and $\vec{v_f}$ are directed downwards. It is standard for shocks that $v_i > v_f$. Consider a non-relativistic particle P in the reference frame of Figure 5 with initial speed V oriented upwards. Explain why, in the absence of other forces, P will continually bounce between both mirrors, and that the speed of P per cycle (i.e. per two collisions, one with each mirror) increases by $2(v_i - v_f)$. You may wish to use diagram(s) to aid your explanation. **[2]**

Kotomi's Magnetic Personality

"We're done with shocks!" Kotomi says happily. "Now, let's talk about magnetic reconnection!"

Magnetic reconnection is the process whereby separate magnetic field lines join together and cause a change in the distribution (or topology, if one wants to be pedantic) of magnetic field lines. This process is exceedingly common in plasmas in space, where magnetic field lines can switch directions often.



Magnetic field lines pushed together

Magnetic field lines meet at centre (boundary)

Reconnection occurs! New field lines move away quickly

Loop happens

Figure 6: Magnetic reconnection process in a nutshell.
Inline arrows dictate magnetic field lines' directions.
Thick arrows represent direction of motion of magnetic field lines.

Generally speaking, there is a region of space where oppositely-directed magnetic field lines are pushed together and meet. This causes the field lines to merge, splitting off to form two new field lines. The new lines 'rebound' in perpendicular directions, much like the snapping of a rubber band.

This has two effects.

1. It creates a 'magnetic vacuum' at the boundary. The original sets of oppositely-directed magnetic field lines are drawn in by the magnetic tension to continue merging at the boundary, forming a looping process.
2. It causes charged particles to accelerate to potentially high velocities, while imparting a significant amount of thermal and kinetic energy to said particles.

The second point is particularly relevant, as the resultant effect is a burst of fast-moving charged particles. This makes magnetic reconnection an important component of various phenomena such as solar flares and geomagnetic storms.

"The acceleration that forms cosmic rays is caused by charged particles repeatedly crossing the boundary," Kotomi lectures. "In a sense, the particles *bounce* between both sides."

(f) Briefly explain TWO possible reasons why a charged particle might repeatedly cross the boundary. **[2]**

Part III A Certain Shocking Accelerator

Akuta then shows up with a sneer. "Feh. Name's Akuta. Pseudonym. I do stuff with vectors. Oi, Kotomi. Ya done with the easy stuff? Now I gotta explain the Fermi rubbish?"

"It's Fermi *acceleration*," Kotomi bites back with a slight frown. "Please treat this seriously, Akuta-san."

"Tch. Whatever."

"Akuta-san!" Kotomi moves to grab his ear. "Do! This! Properly!"

"Ow! Okay, okay! Fine already! Ow! Stop it, you barbarian!"

"Start with cosmic rays!" Kotomi insists sternly. "Got it?"

"I got it already!! Get off! Ow!"

Space Accelerator Fermi

Cosmic rays are high-energy particles moving through space at high relativistic speeds. They primarily consist of nuclei, with a small proportion being electrons and a tiny amount being other particles.

(g) State two astronomical sources of cosmic rays. **[1]**

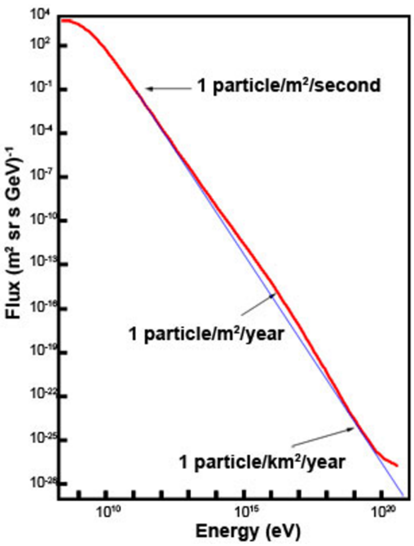


Figure 7: Energy spectrum of cosmic rays.
Thick line: Observed spectrum.
Thin line: Exponential decay line.

The energy distribution, or spectrum, of cosmic rays is a subject of much interest. In general terms, the flux of cosmic rays decreases rapidly as energy of the cosmic rays increases. That is, cosmic rays at lower energies are much more common than cosmic rays at higher energies. The decline (Figure 7)[4] follows a power law with index approximately -2.8.

"The main explanation for this exponential decay is Fermi acceleration," Akuta explains sourly while rubbing his ear. "It's the acceleration of charged particles under multiple reflections."

The original idea of cosmic ray generation, proposed by Fermi, is the reflection of particles off 'magnetic mirrors' multiple times. This concept is, of course, closely linked to the concept of mirrors in shocks. We have seen (cf. Figure 5) that a particle bouncing off mirrors repeatedly can be accelerated.

Magnetic reconnection sites are of particular interest in this case. Cosmic rays can be generated at such sites by a 'basic' version of Fermi acceleration. Such sites are also relatively easy to study using the concepts of shocks, and under shock assumptions the acceleration produces an energy spectrum with index -2.5. It is this simple picture that we will study.

"Simple's good," Akuta grunts with clear irritation at the idea of an overly-complicated picture. "Yeah. Simple's good."

[4] Modified from https://astronomy.swin.edu.au/cosmos/c/cosmic+ray+energies.

It's All About Controlling Vectors

"Here's how we see magnetic reconnection as a shock." Akuta brings up Figure 8.

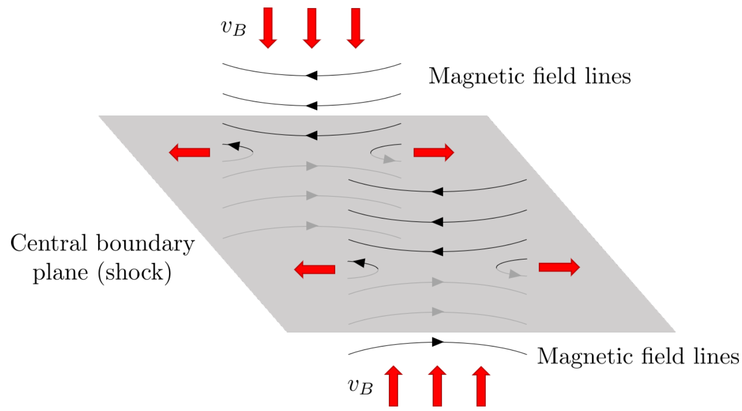


Figure 8: 3D view of magnetic reconnection with planar central shock boundary (cf. Figures 5 and 6). Velocities are taken in the reference frame of the central boundary plane.

In 3D, the central boundary plane at which magnetic field lines meet can be taken to be a stationary shock (cf. Figure 5). The magnetic field lines move, hence the charged particles carried by the field lines move as well. As a consequence, the field lines above and below the central boundary plane are 'magnetic mirrors', and can be taken to be mirrors similar to those introduced in Figure 5. We may assume that the field lines (and hence the particles in the mirrors) are moving towards the central boundary plane non-relativistically at speed v_B each.

(h) A particle P with speed v very close to c in the reference frame of the central boundary plane is approaching the mirror at angle θ to the normal, with initial energy E and magnitude of momentum p. P then collides with the mirror elastically. Let E'' be the final energy of P in the reference frame of the central boundary plane.

(i) **(BONUS)** Show that after the collision, in the frame of the shock, P has final energy

$$E'' = \gamma_B(E' + v_B p'_x),$$

where γ_B is the Lorentz factor with respect to speed v_B, $E' = \gamma_B(E + v_B p\cos\theta)$, and $p'_x = \gamma_B\left(p\cos\theta + \frac{v_B E}{c^2}\right)$. **[2]**

(ii) Show that the fractional energy gain of P per collision is approximately

$$\frac{\Delta E}{E} \approx \frac{2v_B\cos\theta}{c}.$$

(Hint: Recall the relativistic energy and momentum relations $E = \gamma mc^2$ and $p = \gamma mv$.) **[3]**

"So far ya've seen a single particle do stuff," Akuta continues. "But there's more to it. When ya have many particles, they don't all move the same way."

For a large number of particles in three dimensions, it is reasonable to assume that the motion of particles are distributed isotropically. For an isotropic distribution, the probability density[5] $p(\theta)$ of finding a particle at an angle θ to the normal of the central boundary plane satisfies $p(\theta) = 2\sin\theta\cos\theta$.

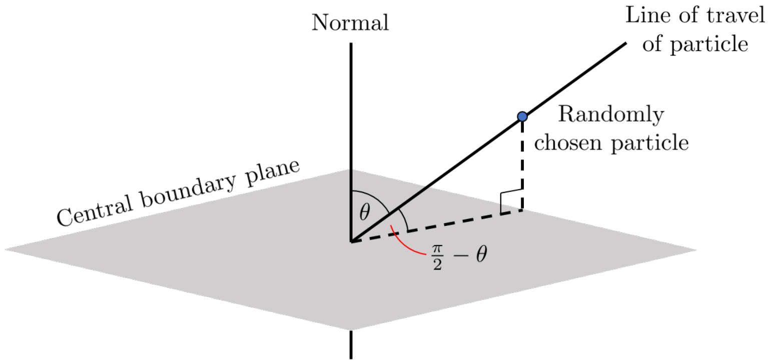


Figure 9: Diagrammatic representation of the angle θ of a particle with respect to the normal. In an isotropic distribution, for any fixed $0 \le \theta_0 \le \frac{\pi}{2}$, the probability density at $\theta = \theta_0$ is $p(\theta_0) = 2\sin\theta_0\cos\theta_0$. Note that since the lines in consideration are undirected, therefore $0 \le \theta \le \frac{\pi}{2}$ always.

(j) For a large number of particles satisfying the speed condition of Part (h), show that the average fractional energy gain per particle per collision is approximately

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4v_B}{3c}.$$

[1]

"Now for the exit." Akuta grins unpleasantly. "They exit at different times. So they don't all got the same final energy."

Ultimately, particles do not stay bouncing between the two mirrors indefinitely. Due to the flux or particle flow, particles must eventually exit the shock region – this is true for all shocks. In the case of magnetic reconnection, charged particles are swept to the side of and away from the boundary by the magnetic field lines moving away from the reconnection site.

(k) Under shock assumptions, the fraction of particles lost per collision is approximately $\frac{2v_B}{c}$. Assume that particle loss is independent of particle energy. Let $N(E)$ be the number of particles accelerated to an energy at least E. Show that

$$\frac{dN}{dE} \approx kE^{-2.5},$$

where k is a constant. **[4]**

"And this," Akuta concludes, "is a power law with index -2.5. Course, the actual cosmic ray picture's a fair bit more complicated." He snorts at the understatement. "But eh. Now you know the basics."

[5] That is, the probability of finding the particle at an angle between θ and $\theta + d\theta$ is $p(\theta)d\theta$.

Epilogue: A Certain Ending Testament

This part serves as the story ending ONLY. There are NO MARKS in this part.

You come away from the tour, your head wildly spinning. You certainly feel the drain on your body! Has your knowledge been thoroughly tested?

Has your brain been thoroughly blended?

As you, Kotomi, and Akuta arrive at the designated ending point of your tour, you see Matou scratching his head sheepishly. Kotomi moves to his side, Akuta following a second later with a small "tch".

"Umm...so...thank you for coming today," Matou says, fidgeting somewhat awkwardly. "We hope you've enjoyed your time today in our city, and we hope you'll consider coming here for your future studies or to research. We hope we've been adequate hosts today and if you have any complaints, please feel free to bring them up with–"

As he continues speaking, you notice Kotomi getting steadily more exasperated.

"Too long, idiot!" Kotomi smacks his head before she turns to you while re-adopting a flowery smile. "I humbly apologise. We thank you and we hope your day has been fruitful. We hope to see you again sometime!"

Then she smirks conspiratorially and adopts a stage whisper. "And if you have any *feedback* about *these two guys...*"

"Ya lookin' for a fight!?" Akuta snarls.

It isn't long before the three are bickering once again.

~ **FIN** ~

 [End of paper]