AstroChallenge 2022
Senior Team Round

Saturday 4th June 2022

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **15** printed pages, including this cover page.

2. Do **NOT** turn over this page until instructed to do so.

3. You have **2 hours** to attempt all questions in this paper.

4. At the end of the paper, submit this booklet together with your answer script.

5. Your answer script should clearly indicate your name, school, and team.

6. It is your responsibility to ensure that your answer script has been submitted.

7. The marks for each question are given in brackets in the right margin, like such: [**2**].

8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

Question 1   Short Answer Questions

An Evening Date

Analyse the story below and answer the following questions.

One clear December evening, Jack brought Jill to a rooftop of a high-rise building to stargaze. Once settled down, Jack started setting up his telescope.

"Help me hold this," Jack said as she passed Jill the OTA.
"What is this?" Jill asked as she reached out to take the OTA.
"It's the optical tube assembly of a refractor telescope, careful not to drop it as the glass inside might crack," Jack explained.

Thereafter, Jack tightened the OTA onto the mount and inserted the eyepiece. He carefully balanced the setup before mounting and aligning the finder scope.

Looking at his watch, Jack realised that it was only 8pm and Comet Leonard was still above the horizon. He quickly pointed his telescope at the comet hoping to catch a glimpse of it before it sets. After 3 minutes of intense searching, Jack exclaimed: "I found it! Come and take a look through the eyepiece!"
"What am I looking at?" Jill questioned with a puzzled look on her face. "Is it that super bright thing near the horizon there?"
"Nope, that bright thing is a planet, it's the brightest planet in the night sky, with a magnitude of 5. But what you are seeing right now is a comet, it has a magnitude of -4." Jack explained patiently.
"Wow! I can't even see the comet in the sky, how were you able to find it and even tell that it is a comet?" Jill asked.

"Well it's quite simple and I shall leave this question as an exercise for the readers." Jack muffled to himself, not wanting Jill to find out that they were just characters in a story.

"What did you say?" "Oh, I was saying I just moved the two axes of the mount up and down, left and right and just happen to point my telescope at it. It's quite simple to use and has no counterweights. Hahahaha." He explained as he tried to hide the awkwardness with laughter.
"I see." Jill mumbled as she stares at the comet as it slowly drifted out of the view of the eyepiece. "Why is the comet moving out? Is it because it is very close to us?"
"It's because the Earth is rotating. The comet itself is quite far, although not as far as the background stars, and the movement is minute," Jack said.

"Let me change to an eyepiece of a shorter focal length so that the comet stays longer in the field of view."
"It's alright, I've seen enough of it." Said Jill as she seemed unimpressed by the comet.

"Alright, let's take a look at the winter night sky instead!" Jack said as he tried to reignite her interest.

(a) **Name and explain three factural errors/mistakes in the above paragraphs.**   **[3]**

(b) **What was the planet near the horizon?**   **[1]**

(c) **What type of mount was used? Explain your answer.**   **[2]**

(d) **Explain how Jack was able to find and differentiate the Comet from other stars.**   **[2]**

(e) **Without using radar ranging, suggest a way that the distance of the comet can be calculated or measured.**   **[2]**

(Story Cont.) After that, the two of them sat down and started to look up into the night sky. Jack explained to Jill how her zodiac constellation looked like and how to recognise the alpha star of that constellation, which is also a red giant. Jack also showed Jill some other deep sky objects (DSOs) through his telescope.

(f) **Name 4 possible DSOs that were seen through the telescope. They must be of different types. Put the type of DSO in bracket beside the name.** **[2]**

(g) **Jill wondered how the red giant in her zodiac constellation was formed. Explain how the red giant was formed.** **[2]**

(Story Cont.) As the night continued, their necks got tired they lied down beside each other and continued their conversation.
"Hey look, it's the moon!" Jack pointed near the horizon at around midnight.
Jill got up and stared at the moon. "Wow it's so beautiful!"

(h) **What was the phase of the moon on that day? Explain your answer.** **[2]**

(j) **Explain how the orbit of the Moon causes the phases** **[1]**

(Story Cont.) At around 3am, they both saw a flash in the far North.
"Was that a shooting star?" She asked.
"Yes, it is the Ursid meteor shower, we are lucky to see one with the moon up being so bright above us." Jack said.
"Make sure to make a wish when you see one again!"

(k) **Around what constellation can the origin of the meteor shower be found?** **[1]**

(m) **Explain how the meteor shower was formed.** **[2]**

(Story Cont.) They stayed up till sunrise before making their way back home. **(Story End.)**

(n) **Bonus: Do you think Jill will go out with Jack again for another stargazing session? Explain your answer.** **[1]**

Question 2   A Celestial Dance

How to Physics?

In Physics, there are generally two ways of solving any question. One is through the analysis of the forces acting on any object and applying Newton's Second Law on the said object. However, this approach is often problematic for complex problem involving multiple forces as it requires dealing with the addition of vectors (which is rather difficult).

The alternative is to analyse the problem by considering the total energy on a system. This approach avoids the complex vector addition that is required when considering forces. Thus, it is often easier to generalise for more complex problems by simply including the energy due to additional objects.

For this question, we will start by looking at the Sun-Earth System and solving it using the Energy approach before generalising it to any arbitrary two-star system. This is often done in analyses that involves binary stars, multiple star systems (such as star clusters) and accurate exoplanet/solar system planet analysis.

Part I   Earth-Sun System

We can start by looking at the total energy of the Sun-Earth-System. Let M be the mass of the Sun and m be the mass of Earth. The total energy for such a system is often written as:

$$E_{total} = \frac{1}{2}m\vec{v}^2 - \frac{GMm}{|\vec{r}_{12}|}$$

where $r_{12} = \vec{r}_S - \vec{r}_E$ and $\vec{r}_S$ and $\vec{r}_E$ are the position vectors of the Sun and Earth respectively with respect to an origin point O. $|\vec{r}_{12}|$ represents the magnitude of the distance between the Sun and Earth whilst $\vec{v}$ is the velocity of the Earth relative to the Sun. **Figure 1** below illustrates the system that has just been described.
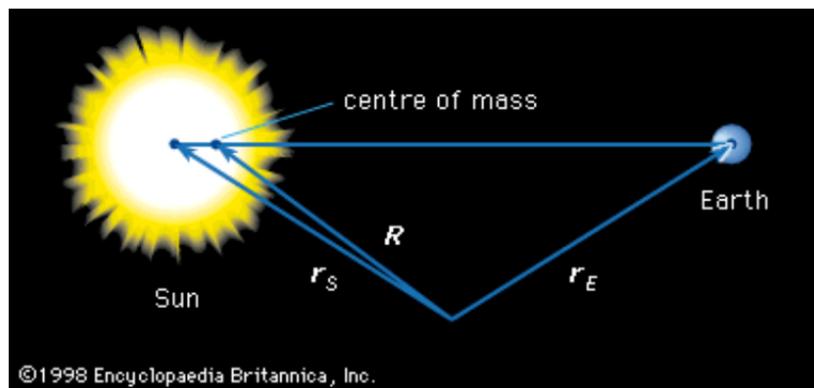

Figure 1: The Sun-Earth System

This total energy expression is written with the assumption that the sun is at the centre of the coordinate system and thus is not moving relative to the coordinate system. However, if we were to generalise this to an arbitrary two-star system, such an assumption **is no longer valid**.

The first step in generalising the approach for general two-star system is by taking an arbitrary coordinate system centre at some origin O as shown in **Figure 1**. In this frame of reference, both the Sun and Earth will have a velocity with respect to the coordinate system and thus an kinetic energy.

(a) **Rewrite the total energy of the Sun-Earth system in this new coordinate system, given the velocity of the Sun with respect to this coordinate frame is $\vec{V}$.**   [1]

Part II   COM Frame

In actual orbital calculations, it is useful to use a particular coordinate system called the Centre of Mass (COM) frame. This is a special frame of reference for multiple-body systems. In this special frame, we are able to separate the often complex forces acting on any object within the system, such as the Earth, into two categories: Internal and External forces. Internal forces due to objects within the system are forces like the gravitational attraction between Sun and Earth, whilst external forces are forces due to objects outside the system of interest (like gravitational force due to other solar system object and due to the Milky Way).

This frame also allow us to analyse the complex motions of objects in this frame into two components: one component is the object's motion around the COM, which is purely due to internal forces and the second component is the motion of the COM (the system itself) due to external forces[1].

With this COM frame, we can now proceed to simplify our expression for the total energy obtained in **(a)**. To do this, we need to first determine the position of the centre of mass. This is given by the relation:

$$mr + MR = 0$$

(b) **Find an expression for r and R in terms of r_{12}. You are given the expression of the reduced mass is:**
$$\mu = \frac{Mm}{M + m}$$

[2]

In general, the value of r, R, and r_{12} can be varying with time but the mass of the objects are constant. For the special case of the two-body system, it is always possible to modify the expression for the total energy such that it is equivalent to having one of the star stationary with the other star moving around the stationary star with a mass equivalent to the reduced mass.

(c) **Show that you can always make such a modification to the two-star system by showing that the total energy can be expressed as the following:**

$$E_{total} = \frac{1}{2}\mu\overrightarrow{v_{12}}^2 - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Extra Info: This is equivalent to the total energy in a coordinate system where one of the stars is stationary and the other star is moving with a speed $|\overrightarrow{v_{12}}|$ where $|\overrightarrow{v_{12}}|$ is the magnitude rate of change of the distance $\overrightarrow{r_{12}}$. **[3]**

(d) **Show that the equation derived in part (c) simplifies to the usual expression for E_{total} when $m << M$. Thus, explain why we can use this simplified expression for the Earth-Sun system as well as any other Sun-(Solar System Planet) pair.** **[2]**

In most orbital questions involving two masses, it is sufficient to further simplify the problem by resolving the motion of the secondary mass into motions in the radial and angular directions as shown in **Figure 2** below.

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[1]An additional benefit of this frame of reference for the two-star system is that it is always centred at a point somewhere in between the two masses, thus reducing the need for complex vector analysis. Thus,from this point on we will use the scale value r and R to denote the magnitude of the vector $\overrightarrow{r_S}$ and $\overrightarrow{r_E}$ with the origin at the centre of mass for simplicity and r_{12} to denote the distance between sun and Earth (Do note that in general for multiple star system, this simplification is not valid)
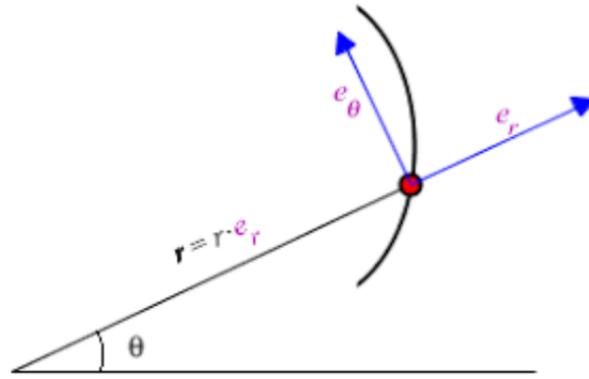
Figure 2: Radial and Angular Components of the Velocity

The radial motion will lead to the secondary mass experiencing periapsis and apoapsis, whereas the angular component together with the gravitational potential adjust according to the radial distance to ensure the total energy is constant, thus maintaining an orbit.

The equation in part (c) can further be simplified by introducing

$$v_{12}^2 = v_\theta^2 + v_r^2$$

Where v_θ^2 and v_r^2 are the angular and radial components of the velocity respectively.

We can define an *effective potential energy* $U_e(\overrightarrow{r_{12}})$ consisting of the angular component of kinetic energy and gravitational potential energy. This will give us the form below.

$$E_{total} - U_e(\overrightarrow{r_{12}}) = \frac{1}{2}\mu v_r^2$$

(e) **Show that the effective potential energy as defined by the above expression is given by:**

$$U_e(\overrightarrow{r_{12}}) = \frac{|\vec{L}|^2}{2\mu|\overrightarrow{r_{12}}|^2} - \frac{G(M+m)\mu}{|\overrightarrow{r_{12}}|}$$

Where $|\vec{L}| = \mu v_\theta |\overrightarrow{r_{12}}|$ is the magnitude of the angular momentum vector, a conserved quantity in this case due to Kepler's Second Law.   **[1]**

(f) **Sketch this effective potential energy with respect to the distance between the two objects. Using the fact that $E_{total} \geq U_e$ for any orbit, explain using the graph where circular and elliptical orbits occur.**   **[3]**

Part III   Restricted Three-Body Problem

Let us shift gears a little. The energy-analysis approach is massively useful as we have seen earlier. However, let us not discount the force-analysis approach. Sometimes, it gives us a more intuitive feel about the problem and how things will play out.

Even just by using only the force-analysis method, we can theoretically solve for a restricted version of the famous *Three-Body Problem.* In this restricted case, we impose a limit where the mass of the third object must be insignificant to the other two masses. We will not be solving it today in this question but rather study its solutions.

For this restricted problem, the solutions that exist are pretty well known. One famous group is the *Lagrange Points* and there are a total of five of them.
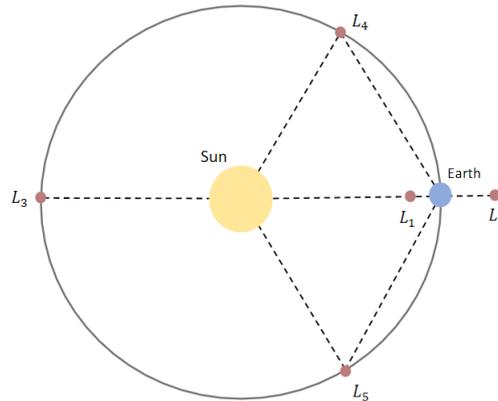

Figure 3: The Five Lagrange points for the Earth-Sun system.

In this case, the location of the points are easier to explain using the idea of balancing forces. They are defined as the points where the gravitational forces from the two large objects and the centrifugal force balance each other. The first three Lagrange points, L_1, L_2, L_3 were in fact discovered by the mathematician Leonhard Euler. It was the final two Lagrange points, L_4 and L_5, which were first discovered by Joseph-Louis Lagrange. That is not to say the energy-approach we did earlier cannot be applied here. The energy-analysis approach is helpful when studying the stability of objects at these Lagrange Points.

(g) **Does the Lagrange Point, L_1, coincide with the Centre of Mass point of the system as per what we defined earlier in part (b)?**   **[1]**

The other point of interest in Physics is that of the Centre of Gravity. For our purposes, we shall define this point to be where the gravitational forces by the two objects negate each other.

(h) **Does the Center of Gravity coincide with the Lagrange point L_1?**
Hence, as well as referencing your answer in part (g), roughly sketch the positions of the Center of Gravity, Center of Mass and L_1 for the Earth-Sun System.
Note: The diagram does not need to be to scale but the order of the three points must be shown clearly.   **[2]**

The other Lagrange point of interest is that of L_2 due to the launch of the James-Webb Space Telescope (JWST) on the 25 December, 2021. After launch, it made its journey towards L_2.

(j) **Show that the Lagrange Point L_2 is $1.51 \times 10^9 m$ away from Earth.**
Hint: You should use a force analysis here. Additionally, you may find that the resultant equation may not be easily solved analytically. You may instead use any numerical methods to show the desired results instead.   **[3]**

(k) **What are the angular diameters of the Sun and Earth at L_2?**   **[2]**

(m) **State one reason why L_2 was chosen as the point to park the JWST?**   **[1]**

Question 3   A Peek at Infinity

Spacetime

The field of Cosmology studies the universe at the largest scale. One part of Cosmology is understanding the evolution of the universe. It uses a few concepts from both Special and General Relativity. One of such concepts is about the idea of Spacetime, which you can think of as the stage for which all events in the universe perform on.

Thanks to the Astronomer Edwin Hubble, we know today that this "stage" is expanding. Understanding the nature of this expansion is thus key to understanding the plausible futures of the universe as well as piece together its history; these we shall soon discover.

Part I   Expansion

Let us first consider a "1+1" Minkowski Spacetime, a spacetime that obeys the *Lorentz* Transformation. In this case, our Minkowski Spacetime will be 1 dimension of flat[2] Space - on the x-axis, with 1 dimension of time on the y-axis. We will use the coordinate system below to represent our spacetime.
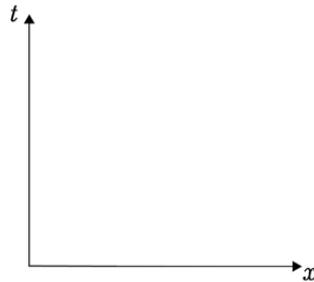

Figure 4: A "1+1" Minkowski Spacetime Diagram

On that diagram, we can label an event which occurs at a *particular location* in space at a *particular time.* This "event" is not limited to abstract ideas. It can also refer to any physical object at a particular location in space at a particular time.

Let us extend this idea. If we plot an event at *multiple* locations and time points, we get a series of dots. By doing this continuously, all the dots get smushed together until we get a line. This line thus represents the trajectory of the event through space *and* time. Remember, the coordinates of a point on this diagram gives you the location and time the plotted event takes place.

For example, the diagram below depicts two **stationary** objects at $x = x_1$ and $x = x_2$ respectively.
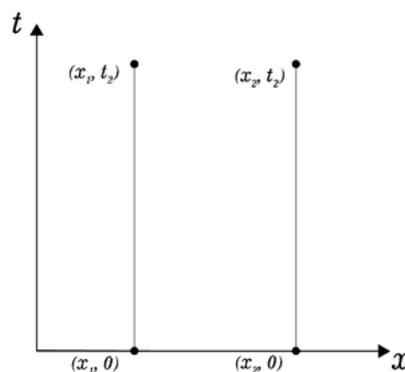

Figure 5: 2 stationary objects on a Minkowski Spacetime Diagram

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[2]Euclidean

For a static universe, space does not expand. In such a non-expanding scenario, the infinitesimal (very small) distance between two points in this coordinate system is given by the following:

$$ds^2 = -c^2dt^2 + dx^2$$

Where c is the speed of light in a vacuum, dt is the infinitesimal (very small) change in time and dx is the infinitesimal (very small) change in the x-coordinate between two points. The following figure illustrates it:
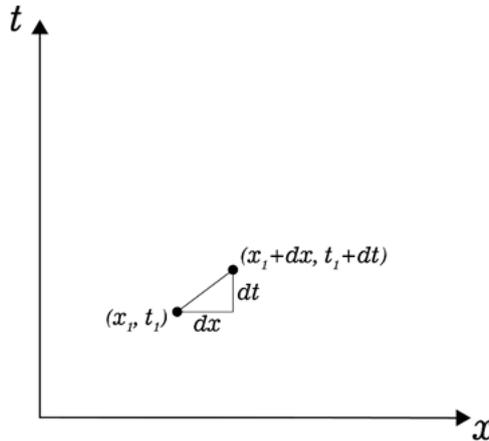

Figure 6: Spacetime interval on a "1+1" static Minkowski Spacetime Diagram

(a) **Taking the case of the non-expanding Minkowski Space, plot the trajectory of an object that moves in the positive x-direction by** b **units with a velocity of** b/t **and then immediately moves back to its original location with the opposite velocity. Label your diagram clearly.** [1]

That was for the case where space was not expanding. In the case of an expanding space[3] however, the infinitesimal distance between two points is now given by:

$$ds^2 = -c^2dt^2 + a^2(t)dx^2$$

This extra $a(t)$ term is what we call the **Scale Factor**. In this case, it is the scale factor of the x-axis and is how we mathematically factor expansion into our spatial dimension.

For two objects pinned to $x = x_1$ and $x = x_2$, when $a(t) = 2$, their physical separation in space has become twice that of when $a(t) = 1$ even though their x-coordinates in our spacetime diagram *did not change*. This is exactly the case we see in the real world, where objects do not move[4] but rather the space between them expands which increases their physical separation.

To expand on this point, a common analogy used to illustrate this point are two marker dots on the surface of an inflating balloon. These dots do not move, but because the balloon is expanding, the two marker dots gets pushed apart.

The scale factor is a *function of time t*. For an expanding universe, that means that $a(t)$ increases with time t. This implies that the distance between two stationary objects with constant x-coordinates in our spacetime diagram will be increasing at $\frac{da}{dt}\Delta x$ where Δx is the distance along the diagram's x-axis between the two objects.

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[3]We shall limit to expansion only in the x-axis.

[4]Because of relativity and the dependency on a reference frame for velocity, you cannot say something "does not move" without also giving the point of reference. But we can still make a physical distinction between movement through Space and the movement due to the expansion of Space. This caveat shall be ignored for our purposes and left as a topic to be discussed on another day.

(b) **Take the case of a *constantly* expanding space. Draw the path that a light pulse takes, as it is emitted from a stationary observer at position $x = x_1$ and time $t = 0$, to a stationary observer at $x = x_2$**
Hint: In a short time interval dt, light travels $a(x)dx$. Will this correspond to a straight line in the diagram? **[1.5]**

This first light pulse that was sent has 10 cycles. Next, an identical light pulse is emitted again from $x = x_1$, just as the tenth cycle of the first light pulse ends at time $t = t_1$.

(c) **On the diagram you have drawn, draw the path that the second light pulse took and compare the time interval between receiving the first and second pulses at $x = x_2$ with t_1. [1.5]**

(d) **Following your answer in part (c), what does this imply about the wavelength of light received at x_2 compared to when it was emitted at x_1?** **[1.5]**

In Cosmology, *Hubble's Law* defines the relationship between the apparent recession velocity, due to expansion, and proper distance (distance between two points in space at the same cosmological time).

(e) **Show that in our "1+1" Minkowski Spacetime, Hubble's Law is given by the following:**

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Hint: The recession velocity is the time differential of proper distance. The proper distance between two points in space at the same cosmological time is given by ds when $dt = 0$. **[2]**

In cosmology, one of the ways we can define the size of the Observable Universe is the maximum distance from where light has had enough time to travel and reach us. To do this, we first must find the age of the Universe.
The age of the Universe as a function of scale factor is given by the following:

$$\Delta t = \int_0^{t(a)} dt$$

Where $t(a)$ is the age of the universe at a particular scale factor a.

From before, we know that the scale factor changes with time. For the sake of our understanding today, we will model the change of the scale factor according to the formula below:

$$\frac{da}{dt} = \frac{\kappa}{a}$$

where κ is a positive constant.

(f) **Using the given information, find an expression for the age of the universe as a function of the scale factor.**
Hint: You can treat da/dt as a fraction and change dt to be the subject of the equation. When doing so, you will need to perform a change in limits from $t(a)$ to $a(t)$ when you perform a change in variable for the integration. **[1.5]**

(g) **From your answer in part (f), conclude whether the model used for the rate of change in the scale factor is consistent with what is happening in our current universe? Explain your conclusion.** **[1]**

Part II   Density

So far what we have done in the previous section is look at **how** the expansion of space can be modelled. This leaves an unsolved question: What gave rise to this expansion?

Our current understanding of the Universe makes use of the Big Bang Model, when space and time came into existence and space has been expanding ever since. In that model, the rate of expansion is governed by the various densities throughout the universe's history.

(h) **While the size of our observable universe is finite, the size of the actual universe itself may well be infinite.**
    Hence, explain what is meant when we say the Big Bang started off from a singularity even though it is impossible for a single point to expand into infinity, given the finite history of the Universe.   **[1]**

As mentioned at the start of this question, this field borrows concepts and understanding from Relativity. From Einstein's General Relativity, mathematician Alexander Friedmann derived equations governing the expansion of the universe. Today we know them as the Friedmann[5] Equations. Equation (1) below is one of the results that Friedmann derived, which governs the rate of expansion of space.

$$\frac{d^2 R}{dt^2} = \left(-\frac{4}{3}\pi G \left[\rho_m + \rho_{rad} + \rho_\Lambda + \frac{3(P_m + P_{rad} + P_\Lambda)}{c^2}\right]\right) \tag{1}$$

Where:
1. R is the scale factor of the universe, analogous to the scale factor a(t) of the x-axis as we have discussed before in the previous section.
2. G is the Universal Gravitational Constant.
3. ρ_m, ρ_{rad}, and ρ_Λ are the equivalent mass density of matter, radiation, and cosmological constant respectively.
4. P_m, P_{rad}, and P_Λ are the pressure exerted by the matter, radiation, and cosmological constant, on space respectively.
5. c is the speed of light in a vacuum.
6. Λ is the cosmological constant.
7. k is a constant that indicates the curvature of space. When $k > 0$, space has positive curvature; when $k = 0$, space has no curvature; and when $k < 0$, space has negative curvature.

In short, equation **(1)** denotes how the acceleration of the scale factor of the universe is affected by the presence of matter, radiation, and the cosmological constant. Matter and Radiation were entirely created from the moment of the Big Bang. The cosmological constant, however, is a property of free space itself. Thus, it is constantly being produced through the expansion of the universe as expansion results in more space.

Now you may be thinking – why would radiation have mass? Aren't photons massless? You would be right, but light carries energy, and with Einstein's famous mass-energy equivalence in the form of $E = mc^2$, we can assign a "mass" to the energy of radiation. We call that the *relativistic mass* of the photons responsible for radiation. This is a parallel to the rest mass that forms most of the mass of matter.

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[5]They are actually also attributed to 3 other people who arrived at them independently: Georges Lemaître, Howard P. Robertson and Arthur Geoffrey Walker. For convenience we shall just refer to them as the "Friedmann equations".

Friedmann had two equations. We have seen the first, now we shall introduce his other equation which outlines the relationship between the equivalent mass density and pressure. This relates the change in equivalent mass (and hence energy) densities as the volume of space increases to the work done by the pressures the various components of our universe exert.

$$\frac{d\rho}{dR} = -3 \left(\frac{\frac{P}{c^2} + \rho}{R} \right) \tag{2}$$

(j) **Given that $\rho_m \propto R^{-3}$, $\rho_{rad} \propto R^{-4}$ and $\rho_\Lambda \propto R^0$, show the following:**

$$P_m = 0$$

$$P_{rad}(\rho_{rad}) = \frac{1}{3}\rho c^2$$

$$P_\Lambda(\rho_\Lambda) = -\rho c^2$$

[3]

(k) **Hence, with reference to equation (1), suggest how the presence of matter, radiation, and the cosmological constant influence the rate of expansion of space.** **[3]**

(m) **With reference to how ρ_m, ρ_{rad}, and ρ_Λ changes with R, arrange the periods when the expansion of the universe was dominated by the influence of radiation, matter, and the cosmological constant in chronological order.** **[2]**

(n) **Why can't the expansion of our current universe be explained by a universe filled with matter and radiation alone?** **[1]**

Question 4   Lights, Camera, Chill

Astrophotography

As the prices of DSLRs drop since their inception, more people now can take photographs of the night sky. The art of Astrophotography has caught on to become a serious pastime for amateur astronomers.

Part I   Photography Jargon

Before we jump in, we should familiarise a little on some technical jargon that is commonly used in the photography community. The ultimate goal of a photographer is to control the amount of light captured by the sensor when taking a photo. It cannot be too much light else the photo will come out overexposed (too bright). The converse is true too, when the camera does not capture enough light.

To achieve a good control over light, photographers concern themselves with 3 main settings:

Camera Settings	What it Controls
Shutter Speed / Exposure Length	The length of time the sensor is exposed to light. (e.g. 1/3s, 1/1000s, 1s, etc.)
ISO (pronounced "eye-soh")	The Sensitivity of the sensor. Higher ISO = More Sensitive
f-Number/Ratio	Aperture size of the lens. Same definition with a telescope's f-ratio.

Table 1: The 3 main camera light control parameters.

These three settings are what a photographer uses to control the amount of light that hits the sensor.

The last setting is known as the Zoom. It controls how magnified the image will be. It is usually expressed as the focal length of the lens. (e.g., 18mm, 35mm, 120mm etc.)

Part II   Camera Properties

James has recently gotten himself an entry-level DSLR camera and wants to try his hands at Astrophotography. He is planning a road trip to a nearby dark-sky site to test what he can possibly capture. The table below shows some basic specifications of the camera he is using:

Camera Property	Value
Model No.	Nikon D3500
Sensor Size (Width × Height)	23.5mm × 15.6mm
Sensor Pixel Resolution (Width × Height)	6000px × 4000px

Table 2: Basic Specifications of James' Camera

James wants to plan his shot, so he decides to understand a bit more of how his camera will work before leaving. He starts by analysing the camera itself.

(a) **If we assume that the pixels in his sensor are squares, calculate the size of each pixel in nanometers. Give your answer to 1 d.p.**   **[1]**

Thinking about it a little more, when James connects his camera lens to the camera, the setup essentially functions like a telescope where the focal length of the scope is equals to that of the lens' focal length.

He wants to know how much of the sky he can take given a certain lens focal length and his sensor size. In essence, he wants to know how big a heavenly body with a certain angular diameter θ will appear on his sensor and thereafter the image taken.

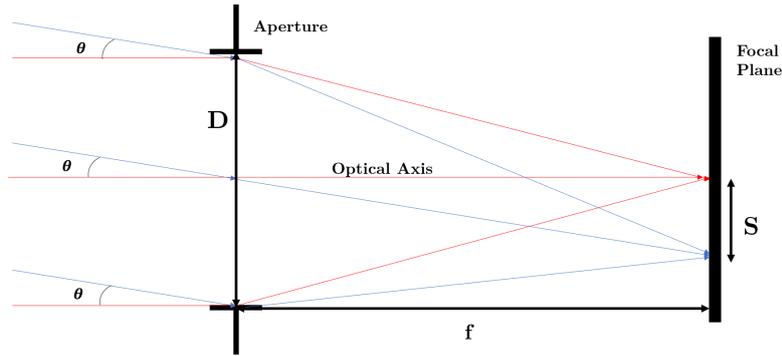In technical terms, he wants to find the *Plate Scale* of his setup.

Figure 7: Plate Scale of a Telescope

(b) **Given an optical setup with a focal length f, show that the Plate Scale (PS), in units of arcseconds/mm, of the setup is given by**

$$PS = \frac{206265}{f}$$

[2]

In this question, for simplicity, we shall treat James' camera setup as something like that of a Newtonian telescope. In reality, the camera lens will change the optics slightly.

The important thing to note in this question is that the lens will act as the Optical Tube Assembly (OTA) of a telescope setup. The lens' focal length will be analogous to that of the OTA's focal length.

(c) **Given that James hooks a camera lens with a focal length of 300.0mm, calculate the Plate Scale of his camera setup.** **[1]**

(d) **If James wants to have the full moon fit just nicely in his image, what focal length does he need?** **[2]**

Part III   Planning the Shot

Because of the low-light level nature of astrophotography, photographers will have to take long-exposures. This works by allowing more photons to hit each pixel on the sensor, the more photons that hit a pixel, the brighter the output from that pixel will be (brighter pixel).

Note that it does not time average the brightness. The brightness of a pixel purely represents the number of photons that hit that pixel when said pixel was exposed to light.

Because of this and the fact that the sky rotates as time passes, a star or any source of light will start leaving trails in the image if the camera is not rotating along with the sky. This is how we get star-trail photos.

James has this sweet zoom lens with a focal length of 300.0mm. He intends to take the photo of the **M11**, the Wild Duck Cluster. It is a rather bright star cluster close to the Celestial Equator. Because James just got into photography, he does not have a tracking mount for his camera.

(e) **James wants to avoid having star trails in his image of M11. What is the maximum amount of time he can leave his sensor exposed before ALL the stars start visibly appearing as star trails in his image? You can treat the stars as point sources in this question.**

   Note: Star trails start becoming visibly apparent when a source that covers 1px stretches out to span 5px in any direction. **[4]**

(f) **If James switches to his 35.0mm lens to get a wide-field shot of the winter night sky, give a brief explanation on whether his maximum exposure time will go up or down.** **[2]**

Part IV   Reality and Noise

In the real world and because of how a sensor works, sometimes a sensor pixel will randomly send an output signal saying it got hit by a photon when it did not. This is the source of *Noise* in a picture and gives rise to a grainy background.

In Astrophotography, noise is much more apparent as we work in a low-light environment and noise can become significant. There are multiple sources where such noise can come from, but today we will only limit ourselves to that of random thermal noise.

We will define the quality of an Astro image using the concept of Signal-to-Noise Ratio (SNR), where the signal (stars) is much brighter than the noise. A higher SNR is a better image in our books. Mathematically, we can define the SNR of an image/group of pixels as follows:

$$SNR = \frac{\text{Total Source Count}}{\sigma_{Noise}}$$

$$\sigma_{Noise} = \sqrt{N_{px} \cdot r \cdot T}$$

Where N_{px} is the number of pixels in the image/group of pixels, T is the exposure length and r is the thermal noise rate of the sensor of **3 counts/px every minute**.

After reading up about noise, James consults his Astronomy teacher who did his thesis on radio and CCD imaging. There, he is introduced the idea of a *Zero-Point Magnitude* (ZPM). In short, it is the measure of the sensitivity of the sensor. Recalling more, James remembered his teacher saying:

"If I say my sensor's ZPM is 21, it means that when I expose my whole sensor to a source of apparent magnitude 21 and allow all the pixels to see it, each and every pixel in the sensor will output 1 count per second.
So when I expose the sensor to a brighter source, the output count rate will increase proportionally with the intensity of the source."

(g) **If James' camera sensor has a ZPM of 16 mag, calculate the SNR for a point-source star of apparent magnitude 10, given that James did a 3-second exposure with his 120mm lens.**   [4]

Part V   Targeting

James has decided to go on his adventure on 30 June 2022. His goal is to capture the bulge of the Milky Way which passes through the constellation of Sagittarius. That night is the night of a New Moon.

The star *Alnasl* (γ Sgr) in the teapot asterism is almost right smack in the middle of the bulge of the Milky Way.

Alnasl Stellar Details	Value
Right Ascension	$18^h07^m16^s$
Declination	$30°25'23.9''$

Table 3: Alnasl Stellar Details

(h) **Given further details that James will be at** $2.540°N, 103.82°E$**, estimate the local civil time when** *Alnasl* **reaches upper culmination. Local Noon occurs at 13:08 local civil time.**

   Note: The location also follows the time zone GMT+8, identical to Singapore.   [4]

$\sim$ **FIN** $\sim$