AstroChallenge 2022
Junior Team Round

SOLUTIONS

Saturday 4th June 2022

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. This paper consists of **33** printed pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your name, school, and team.
6. It is your responsibility to ensure that your answer script has been submitted.
7. The marks for each question are given in brackets in the right margin, like such: [**2**].
8. The **alphabetical** parts (i) and (l) have been intentionally skipped, to avoid confusion with the Roman numeral (i).

Question 1 Short Answer Questions

Part I Chinese Astronomy

The Chinese civilization has had a long history in astronomy, observing the night sky for millennia. In this question, let us dive into history and explore the most important guiding principles and discoveries by the historic Chinese astronomers.

With swathes of fertile land and large populations to support, agriculture was a driving force in the many major developments in Chinese history. With such an emphasis on agriculture, astronomy thus became an important supplementary tool to ensure the continued success of the civilisation as a whole.

(a) **Explain, briefly, the link between astronomy and agriculture.** [1]

Solution:

The Earth's motion around the Sun holds a direct causation to the seasons of the region, thus dictating the most ideal periods to sow and harvest.

Because of the importance of agriculture, the development of the calendar was a crucial and continued effort throughout Chinese history. The Chinese had selected a system of calendar known as the lunisolar calendar, which combined the timings of the cycles of the Moon with the cycles of the Sun. Specifically, the length of each month was determined solely by one cycle of the Moon phases, while the length of the year was approximated by the number of months. However, we know that the number of days a year (approx 365.25 days) is not a perfect multiple of the number of days a month (approx 29.5 days), but one solar year rounds off very crudely to 12 lunar months.

(b) **Assuming that this discrepancy is not corrected for, how many degrees of deviation along its orbit would Earth be from its position on the same lunar date, in a year's (12 months) time? Assume that the Earth's orbit is perfectly circular.** [1]

Solution:

We find the discrepancy between the topical year and the number of days should we only have 12 lunar months:

$$\begin{aligned}\Delta T &= T_{tropical} - T_{12M} = 365.25 - 12 \times 29.5 \\ &= 11.25\end{aligned}$$

This number of days corresponds to and angular displacement along the orbit $\Delta\theta$:

$$\Delta\theta = \frac{11.25}{365.25} \times 360° = 11.1°$$

However, agricultural needs necessitated an effort to correct for this discrepancy, and therefore the leap month (rùn yùe) came into being,which would be inserted at regular intervals in order to help realign the solar and lunar aspects of the calendar.

Ancient astronomers from the *Shang dynasty* had already mathematically deduced an approximate frequency of the leap month, occurring x times every 19 years.

(c) **Find x, the number of leap months that occur every 19 years. You may approximate your numbers to the nearest integer.** **[2]**

Solution:

$$\text{No. of months in 19 years} = \frac{365.25 \times 19}{29.5} = 235.24$$

$$\text{No. of months in 19-"12 months"-years} = 12 \times 19 = 228$$

The difference between the two equals the number of leap months required to sync the two. $N = 235.24 - 228 \approx 7$.

Another major development would be the solar terms (jié qì). These solar terms give a rough guidance to farmers of the typical weather at that point in time to facilitate their agricultural activities. In each solar year, there would be 24 of these solar terms, spread out evenly throughout the solar year.

(d) **Given that the Earth's orbital radius is approximately 150 million km, find the distance along the orbit that Earth has travelled between 2 consecutive solar terms.**
You may give your answer in 3 s.f. **[1]**

Solution:

Every solar term spans: $\frac{360°}{24} = 15°$. Thus the arc length distance between successive solar terms is:

$$S = r\theta = 150 \times 15° \times \frac{\pi}{180°} = 39.3$$

The occurrence of the solar terms are based exclusively upon the position of the Sun in the sky, without relation to the position of the Moon. Some of the most well-known solar terms include the two solstices and two equinoxes. As you may have already known, the December Solstice, also known as the Winter Solstice in the northern hemisphere, is the day in the northern hemisphere when the day is the shortest and the night is the longest, and the sunlight's angle of incidence is the greatest. As such, the Earth in the northern hemisphere receives the least amount of energy from the Sun on that day.

(e) **However, it is a well-known fact that the average temperature in the northern hemisphere is lowest only after the Winter Solstice. Briefly explain why.** **[1]**

Solution:

While Earth receives the minimum amount of energy from the Sun at winter solstice, the temperature of Earth in a particular hemisphere is also determined by the rate of heat loss to surrounding space.

There is a lag in the time it takes for Earth to lose heat to space, hence the lowest temperature is usually only attained some time after winter solstice in January.

Besides agriculture, astronomy was also widely used as a means of making astrological predictions in the old ages due to widespread superstition and belief in microcosm. Astronomy was used by imperial rulers to “foresee” the future in order to make personal and political decisions.

One of the phenomena that was observed with the greatest threat was known as "yíng huò shǒu xīn", which was when Mars came into conjunction with Antares, and appeared to linger in the vicinity of Antares in an almost stationary manner. Such an event was believed to cause widespread chaos across the empire, and endanger the emperor’s political safety.

(f) **What phenomenon would cause Mars to appear almost stationary around Antares? Explain how this phenomenon occurs with modern astronomical knowledge. You are advised to draw a diagram to make your explanation clearer.** [3]

Solution:

Retrogression. It happens when Earth’s tangential velocity perpendicular to the distance to Antares is roughly equal to Mars’ tangential velocity in the same direction. Then Mars appears to be stationary against the background stars from the perspective of the Earth and hence linger around in an almost stationary manner near Antares.

(g) **What naked-eye characteristic that is similar between Mars and Antares could lead the ancient astrologers to fear unrest when the two objects coincided?** [1]

Solution:

Both objects were distinctively deep-red in colour and, at times, similar in brightness.

Other than Mars, another planet which was known since ancient times in China was Venus. Its brilliance caught the attention of ancient astrologers, who believed that it brought blessing and prosperity for the empire. Venus was initially thought as two separate entities, only later discovered to be the same celestial object.

(h) **Explain in a detailed manner why Venus would appear in the night sky usually in the morning or evening only, therefore leading ancient stargazers to believe they were 2 different objects.** [2]

Solution:

Venus’ orbit is inferior to the Earth, i.e. its orbit lies entirely within that of the Earth’s. As such, it will always appear on the same side of the sky as the Sun. (When in eastern elongation, it will appear on the east side of the Sun in the evening, while in western elongation, it will appear on the west side of the Sun in the morning.) Otherwise, it will either be behind the Sun from Earth’s line of sight, or drowned out in sunlight such that it cannot be easily observed.

Another notable phenomenon occurred in 1054, when a very bright star suddenly appeared in the night sky. It was visible in the sky for over 10 months before fading out of view, and for the first 23 days of appearing, it was so bright that it could be observed in broad daylight.
The ancient Chinese astrologers called this a “guest star” (kè xīng), and considered it a sign of prosperity

(j) **What might have been the actual astronomical phenomenon that occurred to bring about this “guest star”? Give an explanation of how it occurs.** [3]

Solution:

The “guest star” was a supernova. As a large star becomes older, it begins to exhaust its fuel and accumulates an iron core which does not undergo nuclear fusion to generate energy for the star. When the core exceeds the mass of $1.44 M_\odot$, gravity causes the core to contract rapidly and leads to an implosion of the entire star. The collapse of the core is then halted abruptly by neutron degeneracy pressure, causing imploding stellar matter to rebound violently and explode in a highly energetic manner, releasing large amounts of energy including light.

(k) **Which 2 types of objects would the previously mentioned phenomenon leave behind that can be observed?** [1]

Solution:

Supernova remnant and Neutron star.

(m) **The red arrow in the figure below shows the location in the sky where the “guest” star appeared. Which constellation was the star located in?** [1]

Solution:

Taurus.

Note: The guest star is the famous SN1054 as mentioned in the next part. It left behind the Crab Nebula (M1).

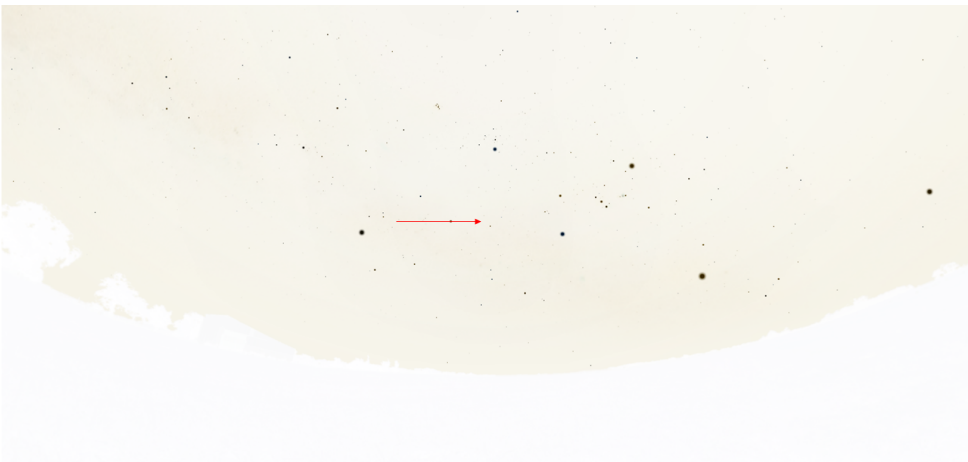

Figure 1: Position of the "Guest Star".

Nearly a thousand year later, astronomers have managed to find what is believed to be the remnant of SN1054, located around 6500 light years (approx. 2000 parsecs) away. Based on historical Chinese records, astronomers estimate that the peak apparent magnitude of the "guest star" was about -6.

(n) **Based on the approximations above, what could have been the maximum absolute magnitude of SN1054?** **[1]**

Solution:

We use the distance modulus:

$$m - M = 5(\lg d - 1)$$

$$M = m - 5(\lg d - 1) = 17.5$$

(o) **Based on your above calculation, how many times brighter than the Sun's current luminosity (absolute magnitude = 4.8) was SN1054?** **[1]**

Solution:

The difference in magnitude was $4.8 - (-17.5) = 22.3$. A 22.3 difference in magnitudes equals to a difference in brightness of:

$$\left[100^{1/5}\right]^{22.3} \approx 2.512^{22.3} = 8.33 \times 10^{8}$$

(p) **Given the observed apparent magnitude of the "guest" star as stated previously, which regular celestial object in the night sky, other than the moon, would be the closest to it in terms of apparent brightness?** **[1]**

Solution:

Venus

Question 2 A Date with the Heavens

Part I Welcome to the Mauna Kāne Observatory

You are an engineer working at Mauna Kāne Observatory, perched on the cone peak of the dormant volcano it is named after, located at 21.39°N, 150.4°W.

Earlier today you were informed that Bob, the resident tour guide, has fallen sick and asked you to welcome two young guests who will be visiting today. Sighing, you stop by the visitor information centre, built on a gentle ascent of the mountain at a mid-level altitude.

A nerdy-looking bespectacled girl and a stocky guy with spiky black hair approach you. College students. "Hi I'm Carole!", the girl beams at you, "and this is Paul." The guy gives you a brief nod and a shy smile. "Let's get started shall we?" you mention while leading them to a Jeep to bring them up the mountain.

A horrendously bumpy jeep ride later, the three of you arrive at the summit, the home of the gargantuan telescopes.

"Man, why they gotta", Paul takes a breath before further moaning, "build them up these high? That was a crazy ride!" Paul takes another breath, arms akimbo before continuing "Is it me or is it getting kind of hard to breath..." "Well, Mauna Kāne is 3950m above sea level." Carole shrugs.

(a) **Suggest why the observatory was build on such a high altitude.** **[1]**

Solution:

There's less atmosphere to see through.

Note: Less light pollution is not the main motivation for building at such high altitudes. It is just a side benefit. There are observatories on hills/mountains with relatively high light pollution due to their proximity to a large city.

"The Mauna Kāne Observatory has 4 identical telescope units here, namely Makalii, Hōkūlei, Kapuahi and Kauluakoko," you introduce. "Each one of them is a reflector, specifically a Nasmyth-Cassegrain, with a massive 12-metre-wide primary mirror. They all operate at visible to mid-infrared wavelengths. (400nm to 10.5μm)."

"Woah that's huge!" Paul exclaims. "But it seems like these big guys are always reflectors? It's different from the one you have right, Carole? What was yours like again?"
"Oh mine is just a simple 5-inch Newtonian." Carole responds.

(b) **Suggest why large telescopes used for astronomical research are almost always reflectors, assuming that budget constraints are not a concern.** **[1]**

Solution:

There are a few plausible reasons for this. Accept any one:

1. Reflectors can work in a wider spectrum of light, as certain wavelengths of light might get absorbed in glass elements found in refractors and catadioptric telescopes.
2. It is easier to maintain cleanliness of a mirror than a lens; a lens requires the entire volume to perfectly free of imperfections and inhomogeneities, but a mirror only requires its surface to be polished.
3. It is not practical to use very large lenses as a lens can only be held at its edges, causing its centre to sag over time due to gravity, distorting the image it produces. Meanwhile a mirror can be supported entirely from underneath.

Part II Telescope Interferometry

"While we mostly use our telescope units separately, we can also use them together as a giant optical interferometer," you mention. "What is an interferometer?" Paul asks.
"It's a bunch of separate telescopes working together as a single telescope to produce higher-resolution images of the stars! We can combine the sources of light from each telescope into a single image." Carole explains.
"Yes! We do that with a technique called Aperture Synthesis. With that, our interferometer can achieve an angular resolution equal to a giant telescope that has the aperture equals to the largest distance between any two component telescopes," you add, showing them a simple schematic sketch of the observatory.

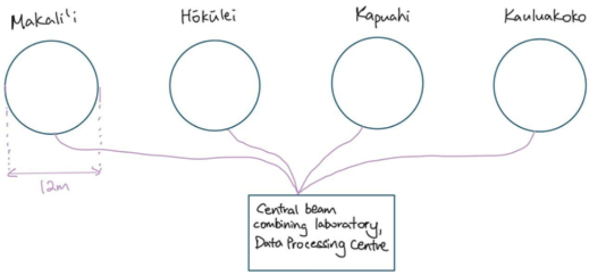

Figure 2: Simplifed sketch of the Mauna Kāne Observatory, showing the four telescope units and the Data Processing Centre. Figure is not drawn to scale.

"Here, you can see that the telescopes are in a line; with Makali'i and Kauluakoko being the furthest from each other. They are spaced 100m apart. The light and data they collect are then combined at our central beam combining laboratory."

(c) **Hence, calculate the theoretical maximum angular resolution due to diffraction of the interferometer at visible light wavelengths, ignoring the effects of atmospheric distortions. Give your answer in arcseconds up to 3 s.f.** [3]

Solution:

For diffraction-limited resolution, we use *Rayleigh's Criterion.* Visible light ranges from 400nm - 700nm. The usual value used is 550nm.

$$\sin\theta = 1.22\frac{\lambda}{d}$$

$$\theta = \sin^{-1}\left[1.22\left(\frac{550\times10^{-9}}{100}\right)\right]$$

$$= 6.71\times10^{-9}\text{rad}$$

$$= 1.38\times10^{-3}\text{arcsec}$$

Note: Due to the range of possible values used for the wavelength of visible light, a range of 1.00657×10^{-3} to 1.7615×10^{-3} were accepted.

Extra Note: Given the small angle of θ, the usage of the small angle approximation was also accepted.

"However, the interferometer does not collect as much light as a single primary mirror with 100 metre-aperture would," you explain.

(d) **Mauna Kāne's total light-collecting are is only a percentage of that of a single 100m aperture primary mirror. Calculate that percentage and express it in 3 s.f.** **[2]**

Solution:

The light collecting area A is:

$$A = N\pi R^2 = 4\pi(6^2) = 452.389m^2$$

A single primary mirror with that aperture of radius 50m will be:

$$A_{single} = \pi r^2 = 7853.98m^2$$

Thus the ratio is:

$$\frac{452.389}{7853.98} \times 100\% = 5.76\%$$

"Does that mean, even though the optical interferometer can pick up finer details, we won't be able to see the dimmer ones?" Carole queries. "Then that doesn't seem very useful..." Paul muses.
"That's not true! Optical interferometers are still hugely helpful. They are the best at observing bright celestial objects. For example, we can use them to capture high-resolution images of stellar surfaces of bright stars like Betelgeuse, observe bright binary star systems."
"Does that mean we can resolve the Double Double fully?"
"Well, let me find that on Wikipedia." Carole pulls out her phone and shows you a table of the orbit pairs in Double Double.

Orbit Pairs in Double Double (Epsilon Lyrae / ϵ Lyr)

	Angular Separation (arcseconds)	Separation (AU)	Remarks
AB - CD	208.2	10500	ϵ_1 - ϵ_2
A - B	2.3	116	Components of ϵ_1
C - D	2.4	121	Components of ϵ_2

Table 1: Orbit pairs of components of the multi-star system Double Double, where ϵ_1 and ϵ_2 are 2 components of the system. ϵ_1 and ϵ_2 themselves are both binary stars.

(e) **State the extent that the Double Double can be resolved by the interferometer at visible light wavelengths, assuming it is producing images at its maximum angular resolution.** **[1]**

Solution:

The interferometer can resolve all components of the Double Double.
The separation between the components is larger than the angular diameter the interferometer can resolve

Part III Atmospheric Turbulence

“Actually, we’re not always able to reach the diffraction limit of angular resolution... The angular resolution that can be obtained is greatly limited by atmospheric seeing due to different refractive indexes of air that light from the stars passes through,” you explain.

"So er, it’s affected by atmospheric turbulence?" Paul asks.
"Yes! Usually, you will expect a telescope with aperture D to be able to resolve objects about λ/D apart based on Rayleigh’s criterion. But our angular resolution is limited to λ/r_0 instead due to atmospheric turbulence, where r_0 is the Fried’s coherence length."

"Fried Chicken?"
"Freed’s coherence length, Paul, it’s pronounced ’Freed’ after the American optics scientist David Fried." Carole corrects him.
"Oops."

Fried’s Coherence Length/Fried Parameter

The Fried parameter is a measure of the quality of optical transmission due to atmospheric turbulence. It has units of length and is typically expressed in centimetres. It is defined as the diameter of a circular area over which the wavefront aberration of light due to passage through the atmosphere is equal to 1 radian.

At a wavelength λ, the Fried Parameter is given by:

$$r_o = \left[\frac{0.423k^2}{\sin\alpha}T\right]^{-\frac{3}{5}}$$

where r_0 is the Fried parameter, α is the altitude of the celestial object being observed, T is the total atmospheric turbulence strength along the path of the starlight through the atmosphere and k is the wavenumber and is $k = 2\pi/\lambda$.

"Huh?" Paul tilts his head as he tries to comprehend.
"Well to put it simply, the worse the atmospheric turbulence, the smaller the Fried parameter. On a night with good seeing conditions, the Fried parameter is about 20cm."

(f) **Hence, state and explain whether Makali’i’s angular resolution would be more affected by atmospheric seeing than Carole’s 5-inch Newtonian.** [2]

Solution:

Makalii’s angular resolution would be more affected by atmospheric seeing. Makalii’s aperture is much larger than the Fried parameter (12m » 20cm).

It can reach an angular resolution due to diffraction much smaller than the angular resolution limited by atmospheric seeing. (i.e., wasted potential if there’s no adaptive optics to overcome the atmospheric seeing!)

A Newtonian with only 5-inch aperture (12.7cm $<$ 20cm) has an angular resolution more limited by diffraction than atmospheric turbulence.

Note: The key thing here is to distinguish that a telescope like Makali’i with a larger aperture has its angular resolution limited by atmospheric seeing and not by diffraction.

"The Fried parameter also tells us about how different it can be viewing a star when it's near the zenith versus when it's near the horizon."

(g) **State how the Fried's coherence length changes as a star's altitude decreases and explain why your answer makes intuitive sense.** **[2]**

Solution:

As the star's zenith angle increases (altitude decreases), the Fried parameter decreases.

Do a common-sense check: As light from a star nearer to the horizon need to travel through more of the Earth's atmosphere to reach the telescope.

"The Fried parameter is also dependent on the wavelength of light we are observing, as we would see." You continued.

$$r_0 = \lambda^b \left[\frac{1.692\pi^2}{\sin\alpha} T\right]^{-\frac{3}{5}}$$

(h) **Show that the equation of the Fried parameter can be expressed in the form above where b is a constant to be determined.** **[2]**

Solution:

By expressing the wave number $k = 2\pi/\lambda$, and taking out λ from the rest of the RHS, we get that $b = 6/5$.

(j) **Hence, state whether a radio interferometer of equal effective aperture is more affected by atmospheric seeing compared to Mauna Kāne's and explain why your answer makes intuitive sense.** **[2]**

Solution:

Mauna Kāne's will be more affected by atmospheric seeing as it observes shorter wavelengths.

As the Fried parameter increases proportionally with an increase in wavelength, the Fried parameter for telescopes operating at longer wavelengths is larger. This makes sense as light of longer wavelengths are less likely to be scattered by the atmospheric particles (as they are larger than the particles).

"Surely there has to be a way to overcome this right...?"

Part IV Adaptive Optics

"Yes, we use adaptive optics to help overcome constraints due to atmospheric seeing. We have a mirror that can deform in real time to reflect corrected wavefronts of starlight!"

You stop to usher them inside the dome-roofed building that hosts Hōkūlei.
"Wow..." Paul reflexively gasp as he stares in awe while Carole claps her hands in delight.
"It sure is cold in here." Carole mentions, turning her clapping into rubbing.
"Well our dome has air-conditioning to keep our telescope constantly cool during the day."

(k) **Explain why the telescope must be kept at constant temperatures.** [1]

Solution:

There are a few plausible solutions, accept any logical answers:

1. The telescope must be kept at constant temperatures to minimise differential rates of expansions and contractions in different parts of a mirror, which will distort the images.
2. The telescope must be kept at constant temperatures to minimise the formation of convection currents, which will affect the refractive indexes of the air in the telescope (and distort the path of light).

(m) **Explain why the telescope must be kept at low temperatures.** [1]

Solution:

The telescope must be kept at low temperatures to minimise infrared/heat noise.

"Man, there's so much going on here," Paul waves at the telescope, "What is what and just what is going on?"

"Well at night when the dome opens up, light will first hit our primary mirror," you point at the enormous mirror, currently covered with a thick canvas to protect it from dust. "Then the light gets reflected to our secondary mirror," you point upwards at a smaller mirror fixed on top with steel beams. "And then onto our tertiary mirror, before into the instruments box," you point at a black circular box affixed at the side of the telescope. "That's where the adaptive optics happens."

"Oh, it looks like a donut. But with only one hole..." says Paul.
"...for the light from the tertiary mirror to enter." Carole adds.
"Hmm, maybe this will make things clearer." You take out a piece of paper and pen and start drawing a diagram of the telescope.

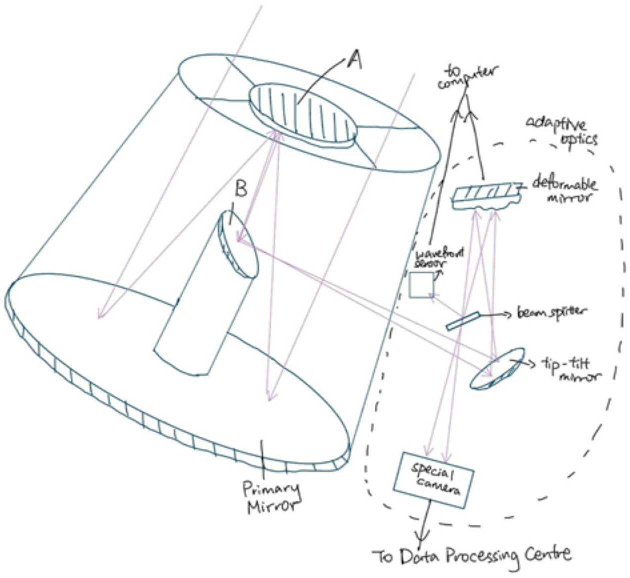

Figure 3: Diagram of the telescope Hōkūlei

"In the instruments box, the light from the tertiary mirror first hits the tip-tilt mirror and gets reflected to our deformable mirror. Our computer deforms the mirror such that its wavefront sensor detects the corrected wavefront of starlight. The beam-splitter splits half of the light from the deformable mirror to our wavefront sensor, and the other half to our special camera, where its shots will be sent to our Data Processing Centre."

(n) **Complete the diagram in *Figure 3* by labelling parts A and B.** **[2]**

Solution:

A: Secondary Mirror
B: Tertiary Mirror

After you finish giving them a more detailed tour, the three of you step off the mezzanine and back out into the open. You bring them to the Data Processing Centre next, where the beam combining laboratory is located.

To your delight, you find your colleague, Alice, staring at a bunch of monitors. One of them shows on screen a collage of grainy black-and-white pictures of stars.

"Yo Alice, we have Paul and Carole here. They're here for a tour. I've already brought them to look at Hōkūlei; mind bringing them around here?"
Alice looks up with a mildly annoyed look. But before she can complain, you have already darted out the door...

Question 3 The Age of Stars

Part I Helmholtz and Kelvin

It was the 1840s. The principle of conservation of energy had just become widely accepted in the scientific community. Soon, a question was asked: what is the source of the Sun's energy?

(a) **Briefly explain how the conservation of energy led scientists to ask this question.** [1]

Solution:

By the conservation of energy, the Sun cannot create energy from nothing and radiate it out. Hence, there must be some mechanism in the Sun that converts stored energy into solar radiation.

Hermann von Helmholtz, a German physician and physicist, proposed that the source of energy is the contraction of the Sun under the influence of its own gravity. To quantify this, we need to understand the energies of orbiting binary bodies first.

Given a system of two bodies with masses m_1 and m_2, orbiting with a constant separation D, its total kinetic energy is K and its gravitational potential energy (GPE) is U.

(b) **By considering the gravitational forces causing the orbiting motion, show that**

$$K = \frac{Gm_1m_2}{2D}$$

[2]

Solution:

The Kinetic Energy (K) is given by:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{1}$$

Because the question stated that the masses are orbiting with a *constant separation* of D, that means both objects are in *circular* orbits about the barycentre. Let d_1 and d_2 be the distances from masses 1 and 2 to the barycentre respectively. Thus $d_1 + d_2 = D$.

In this circular motion, the mutual gravitational attraction provides the centripetal force for both bodies:

$$\frac{Gm_1m_2}{D^2} = \frac{m_1v_1^2}{d_1} = \frac{m_2v_2^2}{d_2}$$

Rearranging:

$$\frac{1}{2}m_1v_1^2 = \frac{Gm_1m_2d_1}{2D^2}$$

$$\frac{1}{2}m_2v_2^2 = \frac{Gm_1m_2d_2}{2D^2}$$

Putting that back into our original equation, we get:

$$K = \frac{Gm_1m_2d_1}{2D^2} + \frac{Gm_1m_2d_2}{2D^2}$$

$$= \frac{Gm_1m_2(d_1+d_2)}{2D^2}$$

$$= \frac{Gm_1m_2}{2D}$$

Note: We can extend this idea to objects that are in elliptical orbits as well, but we replace the separation D. This has been left as an extention and an exercise to the reader.

(c) **If the total energy of the system is E, show that $U = 2E$.** **[1]**

Solution:

The total (mechanical) energy of the system is given by the sum of the Kinetic Energy (K) and Gravitational Potential Energy (U).

$$E = K + U$$

Using the result from **(b)**, we get:

$$E = \frac{Gm_1m_2}{2D} + \left(-\frac{Gm_1m_2}{D}\right)$$
$$= -\frac{Gm_1m_2}{2D}$$

Thus, $2E = U$

Note: Please don't forget that the potential energy term U has the negative sign.

The relationship we have just derived is not only true for this special case, but also for a system of many bodies (averaged over time). It is known as the **Virial theorem**:

$$2\langle E\rangle_t = \langle U\rangle_t$$

Equipped with the Virial theorem, we are ready to understand Helmholtz's idea quantitatively. We will model the Sun as a spherically symmetric mass with a uniform density ρ, total mass M and radius R.

The advantage of assuming spherical symmetry is the applicability of Newton's shell theorem. The theorem pertains the gravitational fields due to spherically symmetric shells of mass. Firstly, the gravitational field outside of a shell is indistinguishable from the gravitational field obtained if all the mass of the shell were concentrated at the centre of the shell. Secondly, the gravitational field strength at every point inside the shell is zero.

(d) **Using Newton's shell method, what is the contribution to the GPE, ΔU, by a *very thin* shell of radius r and thickness Δr?** **[3]**

Solution:

The contribution of GPE by the shell is the additional GPE required to "build" that shell around the existing mass that this new shell will enclose. Let the mass of the shell be Δm and the mass enclosed to be m.
Important to note that this existing mass is spherical and homogeneous with a radius r. We impose this limit for Newton's shell theorem to be applicable.

$$\Delta U = -\frac{GM\Delta m}{r}$$

Given the spherical shape and homogeneous nature of the existing mass, the density must thus be the same everywhere in the mass. Hence:

$$m = M\left(\frac{r}{R}\right)^3$$
$$\Delta m = 4\pi r^2 \times \Delta r \times \rho$$

Thus:

$$\Delta U = -\frac{4\pi\rho GMr^4\Delta r}{R^3}$$

Note: This question has been "simplified" to not invoke the idea of infinitesimals. For completeness and for those interested, we will use it to show the result that follows this subpart.

Again, the change in GPE by adding a infinitely-thin shell is given by:

$$dU = -\frac{GM}{r}dm$$

Following the same logic, we get the same result as before:

$$dU = -\frac{4\pi\rho GMr^4}{R^3}dr$$

Thus, to build a spherically-symmetric mass of radius R, we sum up these changes from $r = 0$ to its final radius $r = R$.

$$U = \int_0^U dU = \int_0^R -\frac{4\pi\rho GMr^4}{R^3}dr$$
$$= -\frac{4\pi\rho GMR^2}{5}$$

Because $\rho = M/(\frac{4}{3}\pi R^3)$, we get the result that follows:

$$U = -\frac{4\pi\rho GMR^2}{5} = -\frac{3GM^2}{5R}$$

Another way to think about this expression is to think of it as the energy liberated when we "build" the object. Think of dU as the energy liberated by "dropping" the thin shell onto the existing mass from infinity.
Thus, if we wanted to "disassemble" the object, we need to put back that same amount of energy. Hence, this expression is also known as the object's *Gravitational Binding Energy*.

In the limit of an infinitesimally small Δr, the expression obtained can be integrated over the entire radius of the Sun, obtaining the GPE of the Sun due to its own gravitational field:

$$U = -\frac{3GM^2}{5R}$$

(e) **Using the Virial Theorem, what is the expression for the total energy of the Sun?** **[1]**

Solution:

Using the Viral theorem, we get:

$$E = \frac{U}{2} = -\frac{3GM^2}{10R}$$

We will assume that the original radius of the newly formed Sun was much larger than the current radius of the Sun. Additionally, assume that the luminosity of the Sun has been approximately constant.

(f) **Use the expression for total energy in (e) to estimate the age of the Sun in Earth years.** **[2]**

Solution:

Let the total energy radiated by the Sun since its formation be ΔE.
Then $\Delta E = |E_{now} - E_{initial}| = |E - E_0|$.
The initial radius was very large, and so we will assume that the system was at $E_0 = 0$. Thus, the change in energy due to contraction is:

$$\Delta E = |E - E_0| = |E|$$
$$= |-\frac{3GM^2}{10R}|$$
$$= 1.137 \times 10^41 J$$

Thus, the Kelvin-Helmholtz timescale T_{KH} is:

$$T_{KH} = \frac{\Delta E}{L_\odot}$$
$$= 2.956 \times 10^{14} s$$
$$= 9.37 \text{ million years}$$

Extra explanations: You might be thinking as to why we did $E_0 = 0$. Look back at the previous subpart. Initially the cloud was very big as compared to the Star. So to model the collapse, we will model this initial state as a cloud that was "infinitely" big. This cloud is already "disassembled" and thus has no gravitational binding energy or $2E_0 = U_0 = 0$.

This age derived is now known as the Kelvin-Helmholtz timescale. The contraction of the Sun was the mainstream theory for the Sun's source of energy for a considerable period of time. However, the theory of evolution by natural selection in Biology emerged and demanded a much longer time for the age of Earth so that the diverse species roaming the world today could exist.

The dispute was settled once and for all when radioactivity was discovered and used to date the age of rocks and minerals on Earth. The oldest minerals were estimated to be 4.2 billion years old.

(g) **Briefly explain how it can be deduced that the contraction of the Sun cannot be the sole source of energy.** **[1]**

Solution:

If the Sun's only source of energy is its contraction, the energy could have only lasted the Sun for roughly 10 million years. Obviously, the age of the Sun cannot be shorter than the age of the Earth. So, the Sun's contraction cannot be its only source of energy.

Another proposed source of energy was chemical reactions. However, chemical reactions involving hydrogen typically release less than $10eV$ of energy per hydrogen atom. It can be shown that a chemical origin for the Sun's energy is similarly implausible.

(h) **With the current mass of the Sun and the assumption that the luminosity of the Sun is approximately constant, estimate for how long chemical reactions can power the Sun (in Earth years).**
Assume that the Sun is composed entirely of hydrogen. **[2]**

Solution:

We find the number of hydrogen atoms in the Sun, N_H:

$$N_H = \frac{M_\odot}{M_H} = \frac{1.989 \times 10^{30}}{1.673 \times 10^{-27}} = 1.189 \times 10^{57}$$

Thus, the total energy liberated from the hydrogen ΔE_H would be:

$$\Delta E_H = N_H \times e_H = 1.189 \times 10^{57} \times 10 \times 1.602 \times 10^{-19} = 1.905 \times 10^{39} J$$

Thus, the chemical timescale T_C would be:

$$T_C = \frac{E_H}{L_\odot} = 1.57 \times 10^5 \text{years}$$

Part II We have to go Nuclear

In the 20th century, two novel theories of physics, Relativity and Quantum Physics, rose into prominence. With them, humanity discovered that an enormous amount of energy is stored in the nuclei of atoms and can be released during fission and fusion. Arthur Eddington (the same scientist who went on an expedition to verify General Relativity) pointed out that nuclear fusion could be a possible source of the Sun's energy. Later, Hans Bethe, a nuclear physicist, proposed *two processes* through which stars carry out their fusion of hydrogen into helium.

(j) **What are the two processes?** [1]

Solution:

Proton-Proton Chain (P-P Chain) and CNO Cycle.

The table below summarises the masses of nucleons and some nuclei.

Nucleon/Nuclei	Mass (u)
Proton	1.007276
Neutron	1.008665
Helium 4	4.002603
Helium 3	3.016029
Tritium	3.016049
Deuterium	2.014102

Table 2: Nucleon/Nuclei and Mass pairings

(k) **Write down an overall equation for the dominant nuclear process happening in our Sun.** [1]

Solution:

The **overall** equation is:

$$4\text{H} \rightarrow {}^{4}\text{He}$$

Note: Leptons are liberated as well but have been omitted in this overall equation.

(m) **How much energy is released per hydrogen atom in the nuclear reaction? Ignore the mass of electrons and neutrinos.** [2]

Solution:

The energy liberated manifested from the binding energy of the nucleus which manifests itself as a mass defect between the constituents and product:

$$\Delta m = (4 \times 1.007275 - 4.002603)(1.660539 \times 10^{-27}) = 4.4006 \times 10^{-29} kg$$

Thus, the energy released per hydrogen is:

$$e_n = \frac{1}{4}\Delta mc^2 = 9.89 \times 10^{-13} J$$

(n) **With the current mass of the Sun and the assumption that the luminosity of the Sun is approximately constant, estimate for how long nuclear fusion can power the Sun (in Earth years).**
Assume the Sun is composed entirely of hydrogen. [1]

Solution:

After all the hydrogen have fused, the total energy liberated is E_N is:

$$E_N = N_H * e_n = 1.176 \times 10^{45} J$$

Thus, the Nuclear Timescale T_N is:

$$T_N = \frac{E_N}{L_\odot}$$
$$= 9.69 \times 10^{10} years$$

This duration estimated is known as the nuclear timescale. However, the scientific consensus is that the nuclear timescale is in fact roughly 10 billion years. So, the Sun is now about halfway through its phase of fusing hydrogen into helium.

(o) **In light of the discrepancy in our estimate of the nuclear timescale in (n), what important consideration did we omit that led to our overestimation of our timescale?** [1]

Solution:

The Sun does not use up ALL its hydrogen for fusion. Only around 10% of the innermost hydrogen ever gets hot enough to undergo fusion.

Although the contraction of stars cannot be their main source of energy, it still plays important roles in certain phases of stellar evolution.

(p) **Suggest two phases of stellar evolution in which the energy from the contraction of stars becomes the dominant energy source.** [1]

Solution:

Before the star becomes a main-sequence star.
When a type of nuclear fuel is used up and the star transitions to fusing a different nuclide.

Note: There are other reasonable answers as well.

Question 4 Great Filters

Globular clusters are amongst the most studied deep sky objects. Understanding them have helped us to confine the age of the universe as well as stellar evolution. Furthermore, their stellar densities have also led many researchers to choose them as probable harbors for life and transmit messages in hope for finding other intelligent life.You are one of the researchers in the field of Astrobiology and the search for extra-terrestrial life (SETI). Today, however, you are on the receiving end.

You received a strange radio message from a distant globular cluster. The transmission originated from the neighboring spiral galaxy *Encanomia*. Below are listed known parameters of the *Encanomia* Galaxy.

Parameter	Value
Right Ascension (RA)	$00^h 42^m 44^s$
Declination (δ)	$+41°16'9"$
Distance	765 kpc
Apparent Magnitude	3.44
Apparent Size	$3.167° \times 1°$
Diameter	67 kpc
Radial Velocity	Negligible
Proper Motion	Negligible

Table 3: Parameters of the Encanomia Galaxy

For all questions below, unless otherwise stated, assume we have all the precise engineering for instruments and measurements.

Part I Globular Clusters

Globular clusters have been extensively used by scientists as "standard candles" to approximate long astronomical distances. By knowing their absolute magnitudes using other methods, we can easily calculate their distance from their apparent magnitudes.

(a) **Propose another alternative method to calculate *Encanomia's* distance.** [1]

Solution:

Other methods include: Cepheid Variables, GCLF, Type Ia Supernovae, RR Lyrae Variables.

You cannot use the following methods: Parallax (Too Far), Tully-Fisher and Hubble's Law (Too Near). *Encanomia* is mentioned to be from the "neighboring spiral galaxy" and its stated distance value puts it within the Local Group. For objects within the Local Group, you cannot use Tully-Fisher or Hubble's Law and expect accurate results.

The Globular Cluster Luminosity Function (GCLF) describes the relation between the bolometric magnitude and number of globular clusters per unit magnitude (ϕ). The equation is established on the assumption that globular clusters have identical absolute magnitude and works best if ϕ is sufficiently large.

$$\phi(m) = Ae^{-(m-m_0)^2/\sigma^2} \tag{2}$$

where m_0 and σ^2 is the turnover (mean) magnitude and variance of the Virgo Cluster respectively.

The distribution of globular cluster within the Virgo cluster is shown below. Note that ϕ is calculated on a 0.2-magnitude interval.

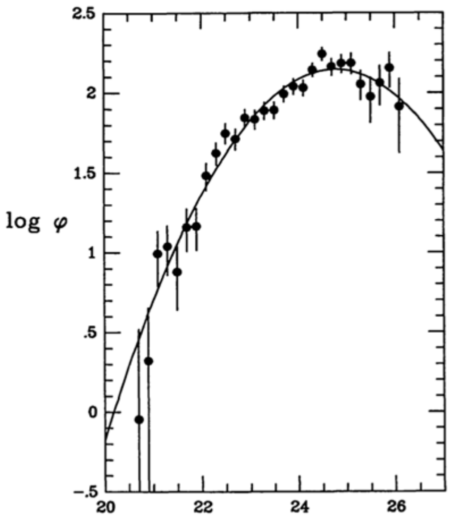

Figure 4: GCLF for the Globular Clusters within the Virgo cluster.
Graph obtained from *Harris et al* (1991)

From the data given, we observe that the turnover magnitude $m_0 = 24.8$ and has a variance $\sigma^2 = 1.4$.

(b) **Calculate the fitting parameter A from equation (1) for the Virgo Cluster.** **[2]**

Solution:

Data points closer to the "peak" are preferable for their low uncertainties. For this solution, we take the $(m, \log\phi) = (24.3, 2.15)$:

$$\phi = A\exp\left\{\frac{(24.3-24.8)^2}{1.4}\right\}$$

$$10^{2.15} = A\exp\left\{\frac{5}{28}\right\}$$

$$A \approx 118$$

(c) **By using your equation from (b), approximate the number of globular clusters within the Virgo Cluster.**
Hint: The area under the curve, like the one in ***Figure 4****, is given by the following Gaussian Integral which has a known general result. Do change the limits of integration as required.*

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

[2]

Solution:

From the definition of ϕ as relative number per unit magnitude, the area under the curve of a $\phi - m$ graph will give us the number. For a continuous sum, we get the expression for relative number as:

$$N = \int_{-\infty}^{\infty} \phi(m) dm$$

Because magnitude can take up vales $m \in (-\infty, \infty)$, that is why we must integrate across that same range.

Thus, plug in the values from **(b)**:

$$N = 118 \int_{-\infty}^{\infty} \exp\left\{-\frac{(m-24.8)^2}{1.4}\right\} dm$$

Some might recognise this immediately as the Gaussian integral and apply the result accordingly. Otherwise, we manipulate it slightly to fit the hint given. First, we observe that it is symmetric about the y-axis, so we can:

$$N = 236 \int_{0}^{\infty} \exp\left\{-\frac{(m-24.8)^2}{1.4}\right\} dm$$

Then, we can use a substitution $x = m - 24.8$ to get:

$$N = 236 \int_{0}^{\infty} \exp\left\{-\frac{x^2}{1.4}\right\} dx$$

This is exactly the form of the hint, where the $a = 1/1.4$. Thus:

$$N \approx 247$$

Part II First Contact

After receiving the transmission, your team noticed immediately that it is an alien message, instead of some natural occurrences. After translating the radio message into readable English, the first part reads out as:

> We hope some brethren out there received our regards;
> Our land is a fair land by a million stars.
> Some are crimson while some are azure, yet all twinkle the same.
> Ours, however, shines bluish-white as its light ebbs and flows.
> Water flows down as when it dims, then fleetingly rises as the season changes.
> Our most arid days twice as bright as our darkest.

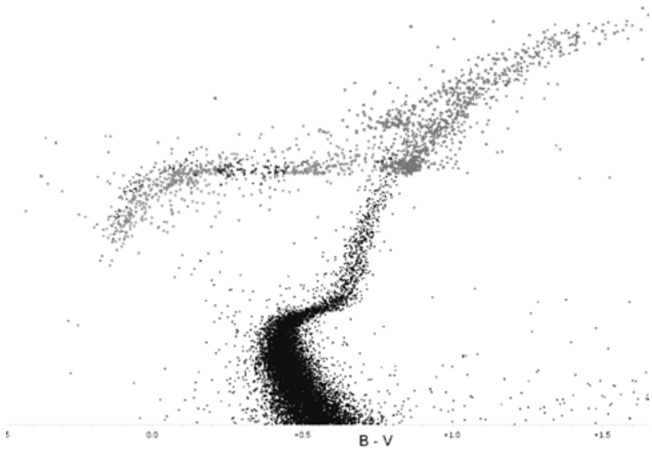

Figure 5: Hertzsprung-Russell Diagram. Image taken from *Wikipedia*

With the known Hertzsprung-Russell diagram (the vertical axis intentionally omitted) of similar globular clusters within the *Encanomia* Galaxy attached above, investigate these lines within the excerpt:

(d) **Hence, approximate the range of its apparent magnitude. Note that the star is "twice as bright" at its brightest compared to its dimmest.
Include any relevant assumptions made.** [2]

Solution:

From the constants sheet, RR Lyrae variables have the absolute magnitude around 0.75.
Ignoring the effect of extinction, we get the apparent magnitude as from the distance modulus as:

$$m - M = 5(\lg d - 1)$$

$$m = 25.2$$

Do note that the star's magnitude varies over time. Thus, we have three plausible cases:

[Case 1] Assume $M = 0.75$ during *maximum*, we get the magnitude at minimum luminosity as:

$$M = 0.75 + 2.5 \lg 2 = 1.5$$

$$m = 25.9$$

Thus the range is $25.2 \leq m \leq 25.9$.

[Case 2] Assume $M = 0.75$ during *minimum*, we get the magnitude at maximum luminosity as:

$$M = 0.75 + 2.5 \lg 0.5 = 0$$

$$m = 24.4$$

Thus the range is $24.4 \leq m \leq 25.2$.

[Case 3] Assume $M = 0.75$ happens somewhere in between minimum and maximum. Thus from the previous part, the range of apparent magnitude must not exceed $[24.4, 25.9]$.

You then explored the implication that an exoplanet can harbor and have harbored intelligent life. Note that for life as we know to exist, said exoplanet must fulfill two criteria:

1. Be on a terrestrial planet (i.e. comprised of minerals and metals)
2. Liquid water is present a majority of the time.

The phrase "water flows down and then rises" may be interpreted as water periodically boiling to vapor briefly before turning back into liquid as the star's luminosity varies.

(e) **If the water on the planet is just about to boil at its hottest season, estimate the distance of the planet from its star.**
You may assume the planet's surface has atmospheric pressure. **[4]**

Solution:

This answer will depend highly on your answer for **(d)** and which case you chose. The solution will use the assumption for Case 1.
The maximum luminosity of the star is:

$$M_{bol} - 4.75 = -2.5 \lg \left(\frac{L}{L_{\odot}} \right)$$

$$L = 1.531 \times 10^{28} W$$

The power absorbed by the planet will then be:

$$P_{in} = \frac{L}{4\pi d^2} \times \pi r^2 = L \left(\frac{r}{2d} \right)^2$$

At thermal equilibrium, the energy absorbed by the planet balances the blackbody radiation radiated out:

$$P_{in} = P_{out}$$

$$L \left(\frac{r}{2d} \right)^2 = 4\pi\sigma r^2 T^4$$

$$d = \frac{1}{4T^2} \left(\frac{L}{\pi\sigma} \right)^{\frac{1}{2}}$$

We use T to be the boiling point of water at 1 atm, thus we obtain $d = 5.27 \times 10^{11} m = 3.52\text{AU}$.

(f) **Is it probable that intelligent life to naturally arise on the planet itself (i.e. not originated from another planet)? Justify your answer.** **[2]**

Solution:

While liquid water is available on the planet, life as we know it can be very difficult to arise due to the implausibility of the planet being terrestrial. Globular clusters are located on the galactic halo, which host stars that have relatively low metallicity. As such, the associated planetary system most likely will not include terrestrial planets.

(g) **If your answer for (f) is "Yes", propose which parts of the galaxy we can likely expect to find other civilizations.**
Otherwise, if your answer for (f) is "No", propose which part of the solar system where we can likely expect people came from. Your answer should be in terms of distance from the Sun. **[2]**

Solution:

Due to the first constraint for intelligent life (planets may be terrestrial), we are more likely to find these civilizations in galaxy parts with high concentration of Population I (metal-rich) stars with stable orbits around the galaxy. In Solar systems of this kind are much more common in **galaxy disks**.

Part III Filters

As you read further onto the message, you noticed the sender's plea for reply:

> Yet our demise is but impending.
> By the while you read this, we have long ceased to be.
> Wars and plague has long ravaged our home.
> For anyone receiving this message;
> we humbly plea for a reply to inform those who come after us that we existed.
> That we have succeeded in finding brethren despite the infinite expanse of the cosmos.

You then decided to send a reply beam – filled with the messages from Earth and their ancestors – back to *Encanomia*. However, since you have a limited chance of transmitting the reply, you must verify the planet's whereabouts accurately.

(h) **By assuming the planet to be Jupiter-sized, find the minimum aperture radius to resolve the planet on visible light. Propose one method that can accommodate such a requirement.** [3]

Solution:

We find the angular size of the planet:

$$\theta = \frac{R_J}{D}$$

$$\theta = 3.028 \times 10^{-15}\text{rad}$$

Note the small angle approximation $\tan\theta \approx \theta$ has been used here.

To minimise the aperture size, we use shortest possible wavelength (violet light of around 400nm) to maximise our resolution. To barely resolve the planet, the minimum resolution must equal the angular size of the planet:

$$\theta = 1.22\left(\frac{400 \times 10^{-9}}{D}\right)$$

$$D = 1.61 \times 10^{8} m$$

Again, the small angle approximation $\sin\theta \approx \theta$ has been used here.

Creating a telescope dish this would be unfeasible. Hence, we utilize *interferometry*; multiple telescope arrays spanning the distance of the required aperture length to accommodate such requirements. Telescope arrays of this magnitude can only be realized on a scale of the Moon-Earth system.

Note: While interferometry in visual bands is challenging and requires extremely precise engineering, they are not impossible.

(j) **Without knowing Encanomia's rotational period, find the ideal diameter for the reply transmitter such that its beam will cover at least the entirety of the galaxy. The reply will be a 75MHz VHF radio beam.** [2]

Solution:

We start by calculating the wavelength of the radio beam:

$$\lambda = \frac{c}{f} = 4m$$

To maximize fidelity of our beam, the ideal beam must be as small as possible while being wide enough to cover the entirety of the galaxy's angular size (3.17°). Hence, from the beam divergence equation, we get:

$$\delta = \frac{4\lambda}{\pi D}$$

$$D = \frac{4\lambda}{\pi\delta}$$

$$D = 9.18m$$

Question 5 Night Sky

Part I The Winter Hexagon

The following diagram is a star chart with the Winter Hexagon marked out. The Winter Hexagon is a hexagonal-shaped asterism visible during winter in most areas except at the southernmost latitudes.

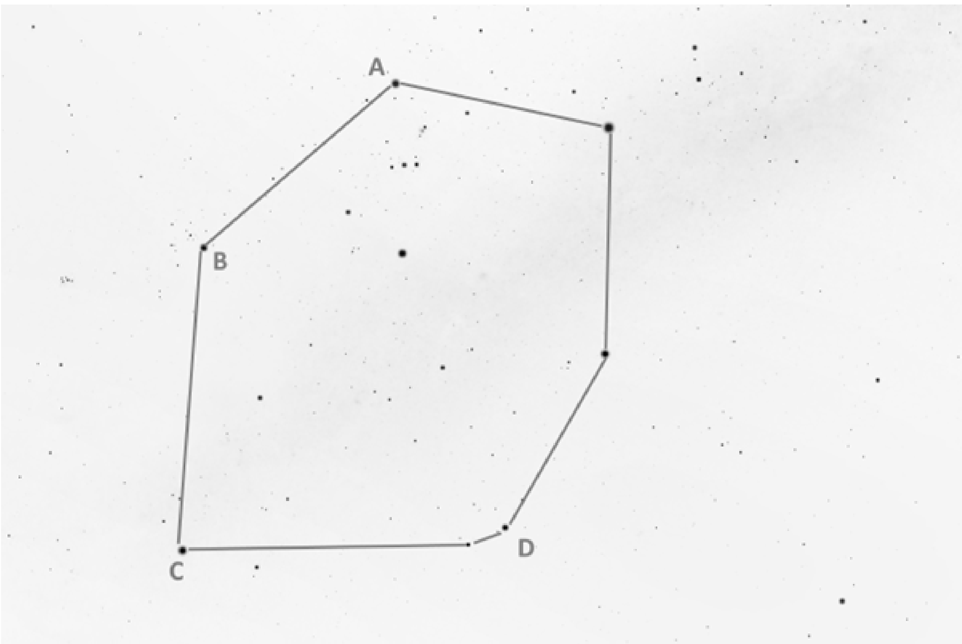

Figure 6: The Winter Hexagon

State the **modern name** for each labelled star in parts **(a)** to **(c)**

(a) **Star A** [1]

Solution:

Rigel (β Ori)

(b) **Star B** [1]

Solution:

Aldebaran (α Tau)

(c) **Star C** [1]

Solution:

Capella (α Aur)

(d) **The star labelled D is *Pollux*. Name the modern IAU constellation that contains star D.** [1]

Solution:

Gemini

Part II Southern Skies

The diagram below represents a view of the night sky right above the horizon in the south. The constellation *Crux*, better known as the Southern Cross, is labelled by **A**. The pointer stars, labelled **B** and **C**, are commonly used with the Southern Cross to determine the location of the South Celestial Pole (**SCP**).

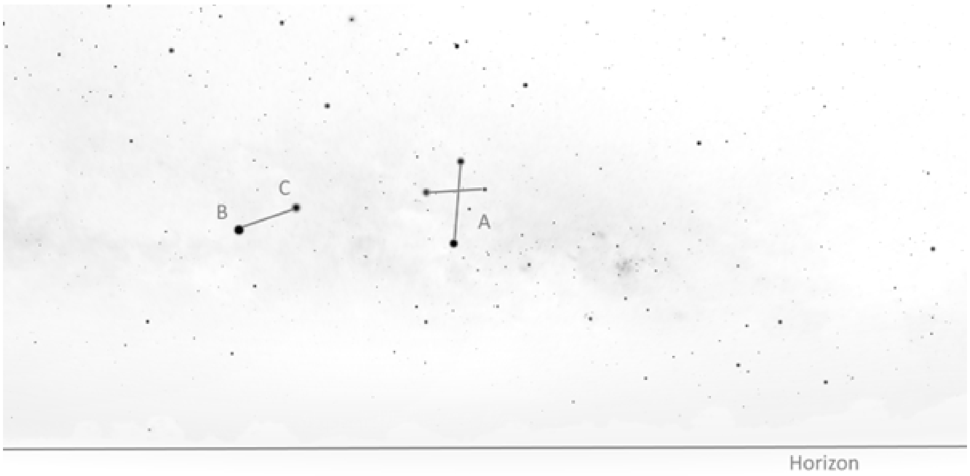

Figure 7: The Southern Sky

(e) **Name star B, which is the brighter of the two pointer stars.** **[1]**

Solution:

Rigel Kent or Rigel Kentaurus. (α Cen)

(f) **Explain how the position of the SCP can be determined with the pointer stars and the Southern Cross.** **[2]**

Solution:

Draw the perpendicular bisector of the line connecting stars B and C. Afterwards, draw a second line from Gacrux down to Acrux and extend it beyond Acrux. The intersection of these two lines marks out the position of the SCP.

(g) **The above method tells us that the SCP is below the horizon in the diagram. Is the current location of the depicted sky located in the Northern or Southern hemisphere?** **[1]**

Solution:

Northern Hemisphere.

(h) **Name the second-brightest star in the night sky. Is it visible in the star chart shown in Figure 7?** **[2]**

Solution:

Canopus. It is **not** visible in the chart given.

Part III Nebulae

A nebula is a giant cloud of dust and gas in space. The following picture (Photo Credit: William Chin, 2004) shows a group of nebulae in the *Orion* constellation.

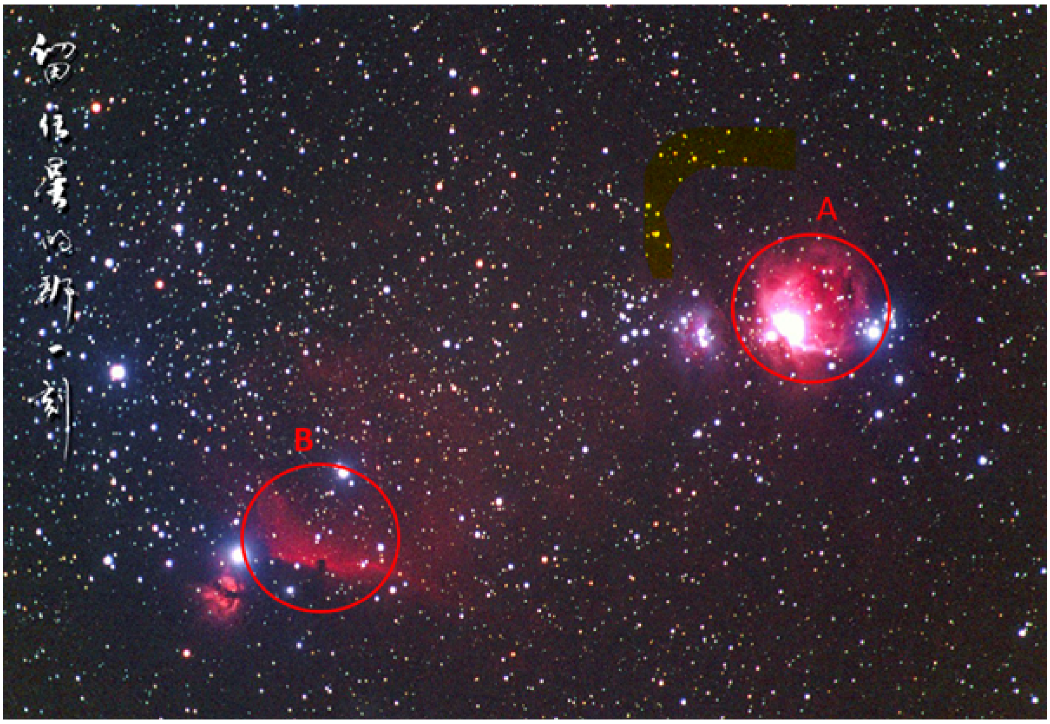

Figure 8: Nebulae in Orion

Given the image in **Figure 8**, answer the following questions:

(j) **Give the names of nebulae *A* and *B* as circled in Figure 8.** **[2]**

Solution:

Nebula A: Great Orion's Nebula (M42)
Nebula B: Horsehead Nebula (Barnard 33)

(k) **State, separately, what type of nebulae are *A* and *B* mainly composed of.** **[1]**

Solution:

Nebula A: Emission Nebula
Nebula B: Dark/Absorption Nebula

Note: M42 is also a Reflection Nebula. Accepted both.

(m) **Explain what are the main differences between each type of nebulae stated in (k)** **[2]**

Solution:

Emission Nebulae emits their own light. Dark Nebulae are dense clouds of dust and gas which absorbs most of the incident visible light from behind the nebula.
Reflection Nebulae reflects light from surrounding stars.

Part IV The Tale of Two Lovers

The following picture is a star chart of the summer night sky.

Figure 9: The Summer Night Sky

The three stars of the summer triangle are associated with the story of the Weaver Girl and the Cow Herder, a romantic legend from Chinese mythology. The Weaver Girl and Cow Herder were separated by the "river of the sky", the Milky Way. On one day every year, the lovers are allowed to cross the "magpie bridge" across the Milky Way to meet with each other. In the night sky, the 2 lovers and the magpie bridge form the Summer Triangle, labeled by **A**, **B** and **C** in the star chart below.

(n) **State the modern names of the stars labelled *A*, *B* and *C* and the respective IAU constellations they each belong to.** **[3]**

Solution:

A: Vega, (α Lyr)
B: Deneb, (α Cyg)
C: Altair, (α Aql)

(o) **As per the legend, which constellation stated in (n) could be the "magpie bridge"?** **[1]**

Solution:

Cygnus

(p) **State another constellation that is fully visible in the star chart above.** [1]

Solution:

Accept any plausible.

Figure 10: The Summer Night Sky Constellations

Note: From the image given above, the following constellations were **NOT** accepted.
Hercules, Draco, Triangulum Australe, Aquarius.

~ **FIN** ~